

# Universality of fluctuations in the dimer model

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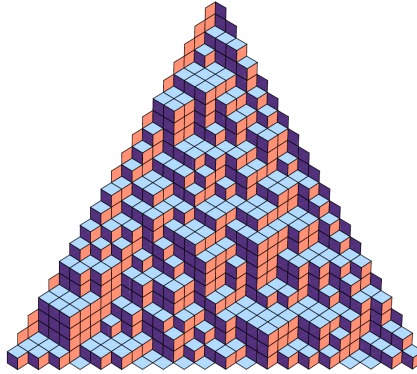
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## Abstract

We present a general result which shows that the winding of the branches in a uniform spanning tree on a planar graph converge in the limit of fine mesh size to a Gaussian free field. The result holds true assuming only convergence of simple random walk to Brownian motion and a Russo–Seymour–Welsh type crossing estimate.

As an application, we prove universality of the fluctuations of the height function associated to the dimer model, in several situations. This includes the case of lozenge tilings with boundary conditions lying in a plane, and Temperleyan domains in isoradial graphs (recovering and extending results of Kenyon [20, 19] and a recent result of Li [32]). In both cases the only assumption on the domain is local connectedness of the boundary.

The proof relies on a connection to imaginary geometry, where the scaling limit of a uniform spanning tree is viewed as a set of flow lines associated to a Gaussian free field. As a result the arguments are very robust, and we discuss several potential applications for which this technique might be useful.



**Figure 1:** Height function of a dimer model (or lozenge tiling) with planar boundary conditions on a triangle. ©R. Kenyon.

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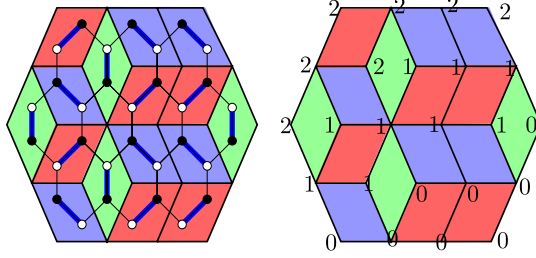
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## 1 Introduction

### 1.1 Main results

Let  $G$  be a finite bipartite planar graph. A dimer covering of  $G$  is a set of edges such that each vertex is incident to exactly one edge; in other words it is a perfect edge-matching of its vertices. The **dimer model** on  $G$  is simply a uniformly chosen dimer covering of  $G$ . It is a classical model of



**Figure 2:** Equivalence of dimer covering and lozenge tiling, and associated height function.

statistical physics, going back to work of Kasteleyn [17] and Temperley–Fisher [42] who computed its partition function. It is the subject of an extensive physical and mathematical literature; we refer the reader to [18] for a relatively recent discussion of some of the most important progress. The popularity of this model stems from its “exactly solvable” character: in particular, a key feature is its determinantal structure ([17]) which can be used to link it to a wide range of topics such as discrete complex analysis, Young tableaux, algebraic geometry.

We can associate to the dimer model a height function which describes a surface in  $\mathbb{R}^3$ , and which contains all the information about the dimer configuration. Hence the dimer model can be thought of as specifying naturally a random function indexed by  $G^\dagger$  (the dual of  $G$ ). A key question in the dimer model concerns the large-scale behaviour of this height function. It is widely believed that in the planar case and under very general assumptions, the fluctuations of the height function are described by (a variant of) the **Gaussian free field**. See Section 2.2 for definitions, but suffice it to say for now that this is perhaps *the* canonical Gaussian conformally invariant process on domains of the complex plane.

In this paper we present a robust approach to the problem of proving universality of the fluctuation of the dimer model on planar bipartite graphs. We now state an example of application of this technique. Consider the dimer model on a finite subgraph of  $\mathcal{H}^{\#\delta}$ , the hexagonal lattice with mesh size  $\delta$ . Then a dimer covering is equivalent to a lozenge tiling as in Figure 2 or equivalently to a stack of cubes of height  $\delta$ . The height function is simply defined as the  $z$  coordinate of the stack at each point. Set  $\chi = 1/\sqrt{2}$  (this is the parameter in imaginary geometry associated with  $\kappa = 2$ , see [34, 35]).

**Theorem 1.1.** *Let  $P$  be a plane in  $\mathbb{R}^3$  and let  $D$  be a simply connected bounded domain in  $P$  with locally connected boundary. Then there exists a sequence of domains  $U^{\#\delta} \subset \mathcal{H}^{\#\delta}$  such that if  $h^{\#\delta}$  is the height function associated to the dimer model (or equivalently the lozenge tiling) of  $U^{\#\delta}$  then  $h^{\#\delta}(\partial U^{\#\delta}) \rightarrow \partial D$  as closed sets in  $\mathbb{R}^3$  in the Hausdorff sense and*

$$\delta^{-1}(h^{\#\delta} - \mathbb{E}(h^{\#\delta})) \circ \ell \xrightarrow{\delta \rightarrow 0} \frac{1}{2\pi\chi} h_{\text{GFF}}^0,$$

*in distribution where  $\ell$  is an explicit linear map determined by  $P$  and  $h_{\text{GFF}}^0$  is a Gaussian free field with Dirichlet boundary conditions.*

The convergence in distribution here is in the sense of finite dimensional marginals when viewed as a process indexed by smooth, compactly supported functions: that is, if we consider a fixed

number of given test functions  $f_1, \dots, f_k$  and call  $X_i = \sum_{v \in U^{\#\delta}} (h^{\#\delta} - \mathbb{E}(h^{\#\delta})) \circ \ell(v) f_i(v)$ , then the  $k$ -tuple of random variables  $(X_i)_{i=1}^k$  converges to the  $k$ -tuple  $((\frac{1}{2\pi\chi} h_{\text{GFF}}^0, f_i))_{i=1}^k$ . In fact, the convergence also holds in distribution on the Sobolev space  $H^{-1-\eta}$  for all  $\eta > 0$ , once  $h^{\#\delta}$  has been extended to a continuous function (essentially by interpolation), see Section 5.2.

As we will see, Theorem 1.1 is an easy consequence of a much more general result proved in this paper and an earlier result of the second author [28]. We now state roughly this general theorem, before explaining further applications to the dimer model. To explain the idea, it is known since the work of Temperley that, on  $\mathbb{Z}^2$ , the height function of the dimer model corresponds to the winding of the branches in a uniform spanning tree. It was observed by Kenyon, Propp and Wilson [26] that this equivalence (known as Temperley's bijection) extends to the superposition of a planar graph and its dual (see below). A more delicate construction due to Kenyon and Sheffield [27] extends this further to general bipartite planar graphs; it is this construction which we rely on here, see [4] for further details.

Hence the general theorem concerns the winding of branches in a uniform spanning tree. We now state this result. Let  $G$  be a planar (possibly directed) graph properly embedded in the plane (so that no two edges cross each other). Let  $G^{\#\delta}$  be the graph obtained by rescaling the plane by  $\delta$ . We assume that  $G^{\#\delta}$  satisfies some natural conditions: these are stated precisely in Section 4.1. The two main assumptions are as follows: (i) simple random walk on  $G^{\#\delta}$  converges to a Brownian motion, and (ii) a Russo–Seymour–Welsh type crossing condition: namely, simple random walk can cross any fixed macroscopic horizontal or vertical rectangle with uniformly positive probability.

Let  $D \subset \mathbb{C}$  be a bounded domain with locally connected boundary. Let  $D^{\#\delta}$  be the graph induced by the vertices of  $G^{\#\delta}$  in  $D$  with boundary  $\partial D^{\#\delta}$  (the precise description is in Section 5.1). Let  $\chi = 1/\sqrt{2}$  as above.

**Theorem 1.2.** *Let  $\mathcal{T}^{\#\delta}$  be a uniform (wired) spanning tree on  $D^{\#\delta}$ , and for any  $v \in D^{\#\delta}$  let  $h^{\#\delta}(v)$  denote the total winding of the branch of  $\mathcal{T}^{\#\delta}$  connecting  $v$  and  $\partial D^{\#\delta}$ . Then*

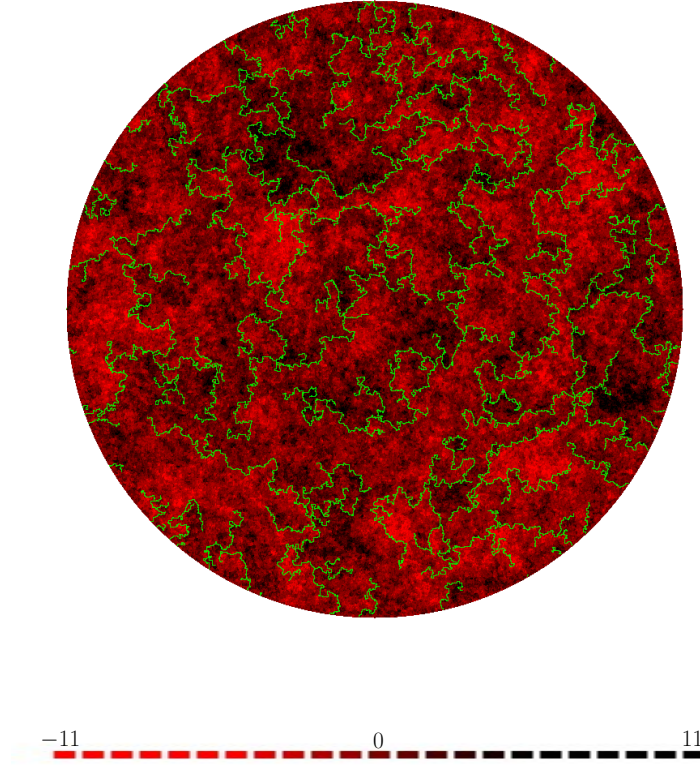
$$h^{\#\delta} - \mathbb{E}(h^{\#\delta}) \xrightarrow[\delta \rightarrow 0]{} \frac{1}{\chi} h_{\text{GFF}}^0$$

*in the sense of distributions, where  $h_{\text{GFF}}^0$  is a Gaussian free field with Dirichlet boundary conditions.*

Here by winding we mean the intrinsic winding, i.e., the sum of the turning angles along the path. See eq. (2.5) for precise definition. Note that the scaling is somewhat different from Theorem 1.1 (there is no renormalisation here) because in that theorem we measure the height defined by lozenges of diameter  $O(\delta)$  whereas here we measure the winding (unnormalised) along paths in the tree.

To get Theorem 1.1 from Theorem 1.2 we use the construction of Kenyon and Sheffield [27] which relates the dimer model on a finite bipartite planar graph (such as a piece of the hexagonal lattice) to a uniform spanning tree on a certain auxiliary directed planar graph called T-graph. The fact that the T-graph satisfy the assumptions in Section 4.1 is proved in [28] and a companion paper [4].

A stronger and more precise form of Theorem 1.2 is stated later on in the paper, see Theorem 5.1 and Theorem 6.1. The full result is stronger for two reasons: first, convergence holds in the stronger sense of convergence in distribution within the Sobolev space  $H^{-1-\eta}$ , as proved in Theorem 5.1. Furthermore, we in fact prove the joint convergence of the winding function *and* spanning tree to a pair (GFF, continuum spanning tree) which are coupled together according to the imaginary geometry coupling, as proved in Theorem 6.1.



**Figure 3:** The winding of a UST on a disc of radius 380 on the square lattice is plotted. The winding is counted with  $\pm 1$  when the branch turns left or right. Some of the branches of the tree are shown in green. This function has (approximately) the law of a GFF on the disc with variance  $\frac{2}{\pi\chi}$ .

## 1.2 Other applications to the dimer model

Theorem 1.2 above has several further applications to the dimer model. To present them it is useful to first recall more details about the relation between dimers and spanning tree. As is customary, when considering the dimer model on a weighted graph, we think of the probability measure

$$\mathbb{P}(\omega) \propto \prod_{e \in \omega} w(e), \quad (1.1)$$

where  $\omega$  is a dimer configuration and  $w(e)$  is the weight of an edge. For the UST on a weighted directed graph we consider the measure

$$\mathbb{P}(\mathcal{T}) \propto \prod_{\vec{e} \in \mathcal{T}} w(\vec{e}), \quad (1.2)$$

where  $\vec{e}$  is an oriented edge,  $w(\vec{e})$  is its weight and the tree is oriented towards a fixed root.

The Kenyon–Propp–Wilson construction of [26] is a simple generalisation of Temperley’s bijection. We start from an arbitrary planar graph  $G_T$  (which will be the graph in which the UST lives) and we define an auxiliary graph  $G$  for the dimers. The dimer graph  $G$  is simply obtained as the superposition of  $G_T$  and its dual, removing a single point from the boundary. The weights on  $G$  are also directly inherited from the weights of  $G_T$ . Then to any dimer configuration  $\omega$  on  $G$  we can associate a spanning tree  $\mathcal{T}$  on  $G_T$  by considering a dimer link to be a half-edge in the tree. This establishes a bijection between wired (oriented) spanning trees on  $G_T$  and dimer configurations on  $G$ , and if  $\omega$  has the law (1.1) then the law of the associated tree  $\mathcal{T}$  is as in eq. (1.2). Finally the dimer height at a point is the winding of the branch from that point in the corresponding tree. See [26] for details.

Using Theorem 1.2 with that construction shows the robustness of our approach. More precisely take  $G_T$  to be any weighted oriented planar graph satisfying our assumptions (Section 4.1), let  $U \subset \mathbb{C}$  be a bounded simply connected domain with locally connected boundary. Let  $G_T^{\#\delta}$  denote  $G_T$  rescaled by  $\delta$ . Let  $U_T^{\#\delta}$  be the sub-graph of  $G_T$  inside  $U$  and let  $U^{\#\delta}$  be the superposition of  $U_T^{\#\delta}$  and its dual. Let  $x^{\#\delta}$  be a point in  $\partial U_T^{\#\delta}$  and  $y^{\#\delta}$  be the vertex of  $U^{\#\delta}$  corresponding to the unbounded face of  $U_T^{\#\delta}$ . Then our result states:

**Theorem 1.3.** *Let  $h^{\#\delta}$  be the height function associated to a dimer configuration on  $U^{\#\delta} \setminus \{x^{\#\delta}, y^{\#\delta}\}$  with law eq. (1.1). As  $\delta \rightarrow 0$ ,*

$$h^{\#\delta} - \mathbb{E}[h^{\#\delta}] \xrightarrow[\delta \rightarrow 0]{} \frac{1}{2\pi\chi} h_{\text{GFF}}^0,$$

*in distribution where  $\chi = (1/\sqrt{2})$  and  $h_{\text{GFF}}^0$  is a Gaussian free field with Dirichlet boundary conditions.*

One nice application concerns the universality of the dimer model on domains of the form  $U^{\#\delta} \setminus \{x^{\#\delta}\}$  within isoradial graphs. More precisely, an **isoradial graph** is a planar graph in which each face can be inscribed in a circle of fixed given radius. When considered with a particular choice of weights, called isoradial weights, an isoradial graph with uniformly elliptic angles satisfies our assumptions by results of Chelkak and Smirnov [7] (the crossing assumption is an easy consequence of Theorem 3.10 in [7]). Hence Theorem 1.3 gives GFF fluctuations for the height function of the dimer model on double isoradial graphs.

A very closely related result was obtained recently by Li [32], see Theorem 1 and Theorem 15 for a more precise statement. The difference between the two setups (beyond the slightly different normalisation of the GFF), is the fact that Li assumes that the boundaries of  $U$  and in fact  $U^{\#\delta}$  contains a straight segment (but no smoothness elsewhere). This type of regularity assumptions are a common feature of approaches based on the inverse Kasteleyn matrix to use techniques from discrete complex analysis (see e.g. also [20]). On the other hand, we do not need to make any such assumption, in fact we only require the boundary to be locally connected which allows for non smooth boundaries.

Another example where the robustness of our approach can be witnessed is the **dimer model in random environment** where  $\mathcal{G}_T$  is taken to be  $\mathbb{Z}^2$  with random weights on the (oriented) edges. A simple example is to sample i.i.d. variables  $p_{\text{vert}}(v) \in [\varepsilon, 1 - \varepsilon]$  symmetric around  $\frac{1}{2}$  for every vertex  $v$  and to consider the walk that jump up or down from  $v$  with probability  $\frac{1}{2}p_{\text{vert}}(v)$

and right or left with probability  $\frac{1}{2}(1 - p_{\text{vert}}(v))$ . Lawler proved that for almost every realisation of the variables, the resulting walk converges to Brownian motion [29] and the other assumptions are trivially verified (in fact the variables do not need to be i.i.d. but only ergodic). When the graph  $\mathcal{G}_T$  is  $\mathbb{Z}^2$ , the domains  $U^{\#\delta} \setminus \{x^{\#\delta}\}$  resulting from the construction described above are called Temperleyan domains. Hence we obtain

**Theorem 1.4.** *Let  $U^{\#\delta} \setminus \{x^{\#\delta}, y^{\#\delta}\}$  be a sequence of Temperleyan domains with the weights inherited from above. Almost surely on the weights, we have*

$$h^{\#\delta} - \mathbb{E}[h^{\#\delta}] \xrightarrow{\delta \rightarrow 0} \frac{1}{2\pi\chi} h_{\text{GFF}}^0,$$

in distribution where  $\chi = (1/\sqrt{2})$  and  $h_{\text{GFF}}^0$  is a Gaussian free field with Dirichlet boundary conditions.

### 1.3 Discussion of the results

**Mean height in dimer models and spanning trees.** Theorem 1.1 describes the limiting distribution of  $h^{\#\delta} - \mathbb{E}(h^{\#\delta})$  and the reader might be interested to know what can be said about the mean itself,  $\mathbb{E}(h^{\#\delta})$ . First, we point out that on the law of large number scale, the mean height of the lozenge tiling is known by a result of Cohn, Kenyon and Propp [8] to converge to a deterministic function which here is simply an affine function (due to our assumptions about the boundary values of the height function).

Our approach yields further information about  $\mathbb{E}(h^{\#\delta})$ . In the spanning tree setting, if  $h^{\#\delta}$  is the winding of branches in a uniform spanning tree (as in the setup of Theorem 1.2) from a fixed marked point  $x$  on the boundary then we obtain

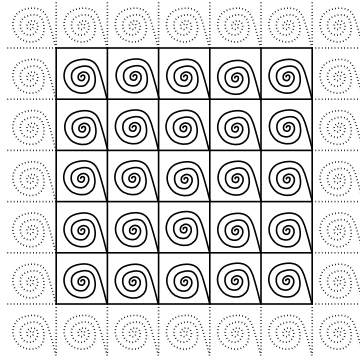
$$\mathbb{E}(h^{\#\delta}) = m^{\#\delta} + u_{D,x} + \frac{\pi}{2\chi}$$

where  $u_{D,x}$  is the harmonic extension of the anticlockwise winding from  $x$  (see eq. (2.9) for a precise definition) and  $m^{\#\delta}$  depends only on the graph and the vertex  $v$  at which we are computing the winding (but interestingly not the domain in which the spanning tree/dimer configuration is being sampled). Note that a consequence of the above mentioned result of Cohn–Kenyon–Propp [8] is that  $m^\delta = o(1/\delta)$  uniformly over the graph; in fact much better bounds can be derived.

For many “reasonable” graphs we suspect that  $m^{\#\delta}$  actually converges to 0, as it is essentially the expected winding of a path converging to a full-plane SLE. Nevertheless some assumptions are clearly needed, as the fact that random walk converges to Brownian motion alone is *not* enough to give control on the mean winding in a UST. For an example of such a situation, see Figure 4. In this example (which consists of the usual square grid to which spiralling decorations have been added at every vertex) it is clear that random walk converges to Brownian motion, but a spanning tree is forced to have some winding at the tip of each spiral, hence the mean winding is biased at the tip of each spiral. This example shows that it is only the **fluctuations** which may be hoped to be universal, while the mean itself will depend strongly on the microscopic details of the graph.

**Relation to a question of Dubédat and Gheissari** The main theorem of this paper, Theorem 1.2, answers a question raised by Dubédat and Gheissari [10], who observed: “Since dimers on  $G$  correspond to Uniform Spanning Trees (USTs) on  $\Gamma$ , and USTs can be generated from LERWs,





**Figure 4:** In this graph, random walk converges to Brownian motion and hence loop erased random walk converges to  $SLE_2$ . Yet each spiral contributes a fixed amount to the winding of a UST.

this suggests universality of dimers when the underlying random walk converges to Brownian motion.” Indeed, this is precisely the point of view which we adopt. We answer this question in the positive with the additional minor assumptions such as the Russo–Seymour–Welsh type crossing estimates, see Section 4.1.

**Relation to earlier results on fluctuations of dimer models.** The study of fluctuations in dimer models has a long and distinguished history, which it is not the purpose of this paper to recall, see [18] for references. However, we mention a few highlights. In [19, 20], Kenyon showed that the height function on the square lattice for Temperleyan domains (for which the boundary conditions are planar of slope 0) converge to a multiple of the Gaussian free field with Dirichlet boundary conditions. The study of dimers on graphs more general than the square or hexagonal lattices was initiated in [25] where they consider tilings on arbitrary periodic bipartite planar graphs. The non-periodic case was first mentioned in [21], also in the whole plane setting.

The interest in the role of boundary conditions was sparked by the observation of the arctic circle phenomenon: for some domains, in the limit the dimer outside of some region (the liquid or temperate region) is deterministic or frozen. This was first identified in the case of the aztec diamond by Jokusch, Propp and Shor [15] (see also the more recent paper [38] by Romik for a different approach and fascinating connections to alternating sign matrices). The case of general boundary conditions for the hexagonal lattice was solved later by Cohn, Kenyon and Propp [8] who obtained a variational problem determining the law of large numbers behaviour for the height function. This variational principle was studied by Kenyon and Okounkov in [24] who discovered that in polygonal domains the boundary between the frozen and liquid regions are always explicit algebraic curves. In this direction we also point out the recent paper by Petrov [36] who obtained convergence of the height function fluctuations to the GFF in liquid regions for some types of polygonal domains.

A paper by Kenyon [22] discusses the question of fluctuations, with the goal of proving convergence of the centered height function to a (deformation of) the Gaussian free field in the liquid region. Unfortunately, the crucial argument in his proof, Lemma 3.6, is incomplete and at this point



it is unclear how to fix it<sup>1</sup>. The issue is the following. The central limit theorem proved in [28] provides an information about convergence of discrete harmonic functions to continuous harmonic functions. However what is needed in [22] is an estimate on the discrete derivative of such functions (i.e., the entries of the inverse Kasteleyn matrix) as well as a control on the speed of convergence so that the errors can be summed when integrating along paths. (There is a more general question here, which is to better understand the links between discrete and continuous harmonic functions on quasi-periodic graphs.) Our work can be seen as a way to get around these issues but more importantly provides a unified and robust approach to the question of convergence of fluctuations.

Finally let us mention that all the above works on fluctuations rely on writing an exact determinantal formula for the correlations between dimers. The main body of work is then to find the asymptotic of the entries of these determinants using either exact combinatorics or discrete harmonic function techniques. Our approach is completely orthogonal, relying on properties of the limiting objects in the continuum rather than exact computations at the microscopic level. This is one reason why the results we obtain are valid under less restrictive conditions on the regularity of the boundary (while such assumptions are typically needed for the tools of discrete complex analysis). In particular, we do not assume the domain to be Jordan or smooth, only to have a locally connected boundary. This is the condition required so that the conformal map from the unit disc to the domain extends to the boundary (Theorem 2.1 in [37]). It is plausible that even this mild condition can be relaxed by appealing to a suitable notion of conformal boundary (e.g., prime ends, see Section 2.4 in [37]) but we did not pursue this here in an attempt to keep the paper at a reasonable length.

## 1.4 Future work

The new approach developed in this paper has the potential to apply to a number of situations which we intend to explore in future works.

A first extension is to more general Riemann surfaces, in particular the torus which has been studied in the dimer case. In an upcoming work [5], we plan to show using similar techniques that the universality of dimer height fluctuation extends to this setting and that the joint law of the limit height field and the tree (which becomes in that setting a cycle rooted spanning forest) is still given by a kind of imaginary geometry coupling. In particular this would extend some recent results on the existence of a scaling limit for the cycles in the uniform cycle rooted spanning forest by Kassel and Kenyon [16] convergence of the full forest and provide an alternative description of this law, as well as prove universality.

A second direction would be to remove from Theorem 1.1 the assumption that the planarity of the boundary conditions. In that setting it is still possible to describe the height function as the winding of the uniform spanning tree in a related graph so we hope to be able to still apply the results of this paper. The main difficulty currently seems to be in the understanding of the scaling limit of random walk on these graphs.

At the continuum level, we note that Theorem 3.2 is of independent interest since it gives an explicit construction of the imaginary geometry map from a uniform spanning tree to a Gaussian free field. We strongly believe that for other values of  $\kappa$ , one can apply the *same* regularisation procedure to a tree of flow lines to recover the associated GFF, or in other word the function from a tree to a field is always the same independently of  $\kappa$  and it consists simply of measuring the

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<sup>1</sup>We thank Fabio Toninelli and Rick Kenyon for helpful discussions regarding this lemma.

“winding” of branches. This suggests a number of results concerning interacting dimers recently introduced by Giuliani, Mastropietro and Toninelli [14]. We believe that if one applies Temperley’s bijection to a configuration of interacting dimers as in [14], the Peano curve of the resulting tree converges to certain space-filling  $\text{SLE}_{\kappa'}$  defined by Miller and Sheffield [35] in these cases and that by adjusting the interaction parameter one can at least obtain any  $\kappa' \in (8 - \varepsilon, 8 + \varepsilon)$ . However it is quite speculative at the moment as we lack tools (like Wilson’s algorithm) to study interacting dimers or its corresponding Temperleyan spanning trees. See [23] for a related question.

## 1.5 Overview of the proof and organisation of the paper

For the convenience of the reader, we summarise briefly the main steps of the proof of Theorem 1.2. We first formulate in Theorem 3.2 a continuous analogue of this theorem, where we study the winding of truncated branches in a continuum wired Uniform Spanning Tree. Branches of this tree are  $\text{SLE}_2$  curves, and therefore a **key idea** is to introduce a suitable notion of (intrinsic) winding. To do so we rely on a simple deterministic observation, see Lemma 2.1 which shows that the intrinsic winding of a smooth curve is equal to the sum of its topological winding with respect to either endpoints. After that, we prove by hand a version of the change of coordinate formula in imaginary geometry:

$$\tilde{h} \circ \varphi - \chi \arg \varphi' = h$$

where  $\varphi : D \rightarrow \tilde{D}$  is a conformal mapping and  $h, \tilde{h}$  are GFF with appropriate boundary conditions in the domains  $D, \tilde{D}$ . This equation is taken as the starting point of the theory of imaginary geometry (see e.g. [34, 41]) but here it must be derived from the model and our definition of winding. Together with the domain Markov property of the GFF and of the continuum UST (inherited from the domain Markov property of SLE), this implies that the winding of a continuum UST is a Gaussian free field with appropriate boundary conditions.

After Theorem 3.2 is proved, we return to the discrete UST, and we write

$$h^{\#\delta} = h_t^{\#\delta} + \epsilon^{\#\delta} \tag{1.3}$$

where  $h^{\#\delta}$  is the winding of the branches of the discrete tree,  $h_t^{\#\delta}$  is the winding of the branches truncated at capacity  $t$ , and  $\epsilon^{\#\delta}$  is the difference. When  $t$  is fixed and  $\delta \rightarrow 0$  there is no problem in showing that  $h_t^{\#\delta}$  converges to the regularised winding of the continuum UST (this essentially follows from results of Yadin and Yehudayoff [43]). By Theorem 3.2 mentioned above, we also know that as  $t \rightarrow \infty$ ,  $h_t$  converges to a GFF. It remains to deal with the error term  $\epsilon^{\#\delta}$ . The main idea for this is to construct a multiscale coupling with independent full plane USTs, which relies on a modification of a lemma of Schramm [39]. This allows us to show that the terms  $\epsilon^{\#\delta}$  from point to point have a fixed mean and are independent of each other, even if the points come close to each other. This is enough to show that when we integrate against a test function, the contribution of these terms will vanish. In order to do so we need to evaluate the moments; however this requires precise a priori bounds on the moments of the discrete winding. We therefore first derive a priori tail estimates on the winding of loop-erased random walks, and this is where we make use of our RSW crossing assumptions.

**Organisation of the paper.** The paper is organised as follows. In Section 2, some background and definitions are provided. In Section 3 we formulate and prove the continuum analogue of Theorem 1.2, Theorem 3.2. In Section 4 we derive the required a priori estimates on winding

and describe the multiscale coupling. We put all those ingredients together in Section 5, which completes the proof of Theorem 1.2.

Throughout the paper,  $c, C, c', C'$  etc. will denote constants whose numerical value may change from line to line. Throughout  $\text{Arg}$  will denote the principal branch of argument with branch cut  $[0, -\infty)$ . Also all our domains are bounded unless explicitly stated.

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## 2 Background

### 2.1 Schramm–Loewner evolution

In this section, we are going to provide some background material on SLE or Schramm–Loewner evolution. SLE is a family of conformally invariant random curves in the unit disc  $\mathbb{D}$  which are supposed to describe several aspects of statistical physics models. In this paper we are concerned with the radial version of SLE, see [30] for more details. Radial SLE in a domain  $D$  with parameter  $\kappa \leq 4$  starting from a point  $a \in \partial D$  and targeted towards  $z \in D$  is described by a family of curves  $\gamma[0, t]$  for  $t > 0$  where  $t$  is the capacity defined by  $e^{-t} = R(z, D \setminus \gamma[0, t])$  (we call this parametrization by capacity).  $\text{SLE}_\kappa$  in the unit disc  $\mathbb{D}$  starting from 1 and targeted to 0 can be defined via the (unique) family of conformal maps  $g_t : \mathbb{D} \rightarrow \mathbb{D} \setminus g_t$  with  $g_t(0) = 0$  and  $g'_t(0) > 0$  which satisfy the following differential equation (called the radial Loewner equation) for each  $z \in \mathbb{D} \setminus \gamma_t$ :

$$\frac{\partial g_t}{\partial t} = g_t(z) \frac{e^{i\sqrt{\kappa}B_t} + g_t(z)}{e^{i\sqrt{\kappa}B_t} - g_t(z)}; \quad g_0(z) = z \quad (2.1)$$

$\text{SLE}_\kappa$  enjoys conformal invariance hence one can obtain radial  $\text{SLE}_\kappa$  curves in other domains for other starting and target points simply by applying conformal maps to the curve described by (2.1). It is worthwhile to note here that radial  $\text{SLE}_\kappa$  curves targeted to 0 is symmetric under reflection.

**Notation.** For  $z \in D$ , we denote by  $R(z, D)$  the conformal radius of  $z$  in the domain  $D$ . That is, if  $g$  is any conformal map sending  $D$  to the unit disc  $\mathbb{D}$  and  $z$  to 0, then  $R(z, D) = |g'(z)|^{-1}$ .

### 2.2 Gaussian Free Field

We include a few lines of definition of the GFF mostly to fix normalisation. See [40] and [3] for more thorough definitions. Let  $D$  be a domain in  $\mathbb{C}$ . Let  $G_D$  be the Green function in  $D$  (with Dirichlet boundary conditions), i.e.,  $G_D(x, y) = \int_0^\infty p_t^D(x, y) dt$  where  $p_t^D$  is the transition kernel for a Brownian motion killed upon exiting  $D$ . The Gaussian free field, viewed as a stochastic process indexed by test functions  $f \in C^\infty(\bar{D})$ , is the centered Gaussian process such that  $(h, f)$  is a normal random variable with variance  $\iint G_D(x, y) f(x) f(y) dx dy$ .

Alternatively, if  $f_n$  is an orthonormal basis for the Sobolev space  $H_0^1(D)$  induced by the Dirichlet inner product

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f \cdot \nabla g, \quad (2.2)$$

then  $h = \sum_n X_n f_n$  where  $X_n$  are i.i.d. standard real Gaussian random variables. As a consequence of Weyl's law, it can be seen that this sum converges almost surely in the Sobolev space  $H^{-\eta}$  for any  $\eta > 0$ . (In fact, when  $\eta > 1$  the series converges absolutely a.s.). Note that even though the GFF is not defined pointwise we will sometimes freely abuse notations and write  $\mathbb{E}(h(x_1) \dots h(x_k))$  for its  $k$ -point moments (which are unambiguously defined if the  $x_i$  are distinct).

Finally, for any function  $u$  on the boundary of  $D$ , a GFF in  $D$  with boundary condition given by  $u$  is defined to be a Dirichlet GFF in  $D$  plus the harmonic extension of  $u$  in  $D$ . (In fact this also makes sense when  $u$  is rougher than a function, provided that it can be integrated against harmonic measure).

## 2.3 Topological and intrinsic winding of curves

The goal of this section is to make precise several notions of windings of curves which we use in this paper. A self avoiding curve is an injective continuous map  $\gamma : [0, T] \mapsto \mathbb{C}$  for some  $T \in [0, \infty]$ . Since we will work with quantities that do not depend on the parametrization, we assume in this section that  $T = 1$ .

**Topological winding:** The *topological* winding of a curve *around a point*  $p \notin \gamma[0, 1]$  *taken continuously* is defined as follows. We can write

$$\gamma(t) - p = r(t)e^{i\theta(t)} \quad (2.3)$$

where the function  $\theta(t) : [0, \infty) \mapsto [0, \infty)$  is taken to be continuous. We define the winding of  $\gamma$ , denoted  $W(\gamma, p)$  or  $W(\gamma[0, 1], p)$ , to be  $\theta(1) - \theta(0)$ . For an interval of time  $[s, t]$ , note that  $\gamma$  restricted to  $[s, t]$  defines a curve whose winding is simply  $\theta(t) - \theta(s)$ . We write with a slight abuse of notation  $W(\gamma[s, t], p) = \theta(t) - \theta(s)$ .

We can extend the above definition to  $p = \gamma(0)$  or  $p = \gamma(1)$  by the following formula when they make sense :

$$W(\gamma, \gamma(1)) = \lim_{t \rightarrow 1} \theta(t) - \theta(0) \quad W(\gamma, \gamma(0)) = \theta(1) - \lim_{s \rightarrow 0} \theta(s). \quad (2.4)$$

**Intrinsic winding:** The notion of intrinsic winding we describe now is perhaps a more natural definition of windings of curves. This notion is the continuous analogue of discrete windings of loop erased paths in  $\mathbb{Z}^2$  which can be defined just by the number of anticlockwise turns minus the number of clockwise turns. Notice that we do not need to specify a reference point with respect to which we calculate the winding, hence our name “intrinsic” for this notion.

We call a curve smooth if the map  $\gamma$  is smooth (continuously differentiable). Suppose  $\gamma$  is smooth and  $\forall t, \gamma'(t) \neq 0$ . We write  $\gamma'(t) = r_{\text{int}}(t)e^{i\theta_{\text{int}}(t)}$  where again  $\theta_{\text{int}}(t) : [0, \infty) \mapsto [0, \infty)$  is taken to be continuous. Then define the intrinsic winding in the interval  $[s, t]$  to be

$$W_{\text{int}}(\gamma, [s, t]) := \theta_{\text{int}}(t) - \theta_{\text{int}}(s). \quad (2.5)$$

The total intrinsic winding is again defined to be  $\lim_{t \rightarrow T} W_{\text{int}}(\gamma, [0, t])$  provided this limit exists. Note that this definition does not depend on the parametrisation of  $\gamma$  subject to smoothness and the derivative being non zero.

The following topological lemma connects the intrinsic winding of a curve with its topological winding around its endpoints. In words, the intrinsic winding of a curve is the sum of the topological winding seen from both its endpoints.

**Lemma 2.1.** *Let  $\gamma[0, 1]$  be a smooth self avoiding curve with  $\gamma'(s) \neq 0$  for all  $s$ . We have*

$$W_{\text{int}}(\gamma) = W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)). \quad (2.6)$$

*Proof.* Let  $x [2\pi]$  denote  $x$  modulo  $2\pi$  for any real number  $x$ . Recall that  $\text{Arg} \in (-\pi, \pi]$  is the principal branch of argument with branch cut  $(-\infty, 0]$ . Note that

$$\lim_{u \rightarrow t, u < t} \text{Arg}(\gamma(u) - \gamma(t)) \equiv \pi + \text{Arg}(\gamma'(t)) [2\pi], \quad \lim_{u \rightarrow s, u > s} \text{Arg}(\gamma(u) - \gamma(s)) \equiv \text{Arg}(\gamma'(s)) [2\pi].$$

where  $x[2\pi]$  is  $x$  modulo  $2\pi$ . Therefore for any  $t$ ,

$$\begin{aligned} W(\gamma[0, t], \gamma(t)) + W(\gamma[0, t], \gamma(0)) &\equiv (\pi + \text{Arg}(\gamma'(t)) - \text{Arg}(\gamma(0) - \gamma(t))) \\ &\quad + (\text{Arg}(\gamma(t) - \gamma(0)) - \text{Arg}(\gamma'(0))) [2\pi] \\ &\equiv \text{Arg}(\gamma'(t)) - \text{Arg}(\gamma'(0)) [2\pi] \\ &\equiv W_{\text{int}}(\gamma[0, t]) [2\pi]. \end{aligned}$$

Equivalently we can write

$$W(\gamma[0, t], \gamma(t)) + W(\gamma[0, t], \gamma(0)) = W_{\text{int}}(\gamma[0, t]) + 2\pi k_t \quad (2.7)$$

for some  $k_t \in \mathbb{Z}$ . However, as  $t$  goes to 0, both  $W(\gamma[0, t], \gamma(t)) + W(\gamma[0, t], \gamma(0))$  and  $W_{\text{int}}(\gamma[0, t])$  go to 0 which implies  $k_0 = 0$ . Since  $\gamma$  is smooth and self avoiding, it is easy to check that both the winding terms are continuous in  $t$ . But this implies  $k_t$  is continuous in  $t$  and hence we conclude  $k_t = 0$  for all  $t$ . This completes the proof.  $\square$

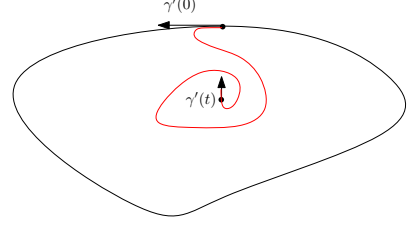
**Remark 2.2.** The relation (2.6) is significant for our analysis of winding of curves which are rough. Typically the curves will be non rectifiable (like SLE curves) or piecewise smooth discrete paths. In both these cases, only the topological winding is well defined. We will therefore use eq. (2.6) to emulate the intrinsic winding for curves where it cannot directly be defined.

A further important fact is that the topological winding around a boundary point is only a function of the domain.

**Lemma 2.3.** *Let  $D$  be a simply connected domain and let  $x$  be a fixed boundary point. Let  $\gamma$  be a curve in  $\bar{D}$  starting at  $x$  (not necessarily self avoiding) and assume that  $\gamma$  is smooth near  $x$  and that  $\gamma$  does not visit  $x$  after time 0. We have*

$$W(\gamma, x) = \arg_{x-D}(x - \gamma(1)) - \text{Arg}(-\gamma'(0)),$$

where  $\text{Arg}$  is the principal value of argument taken with branch cut  $(-\infty, 0]$  and  $\arg_{\gamma(0)-D}$  is the version of the argument taken in  $\gamma(0) - D$  (which does not contain 0 since  $\gamma(0) \in \partial D$ ) so that the arg matches with  $\text{Arg}$  in the direction of  $-\gamma'(0)$ .



**Figure 5:** An example illustrating eq. (2.8). The intrinsic winding of the curve is  $-5\pi/2$  and the topological winding around  $\gamma(t)$  is  $-3\pi$  and around  $\gamma(0)$  is  $\pi/2$ . Here  $\arg_{\gamma(0)-D}(\gamma(0) - \gamma(t)) = \pi/2$  and  $\text{Arg}(-\gamma'(0)) = 0$ .

*Proof.* Clearly as above we have for all  $t$ ,

$$W(\gamma(0, t), x) \equiv \text{Arg}(\gamma(t) - x) - \text{Arg}(\gamma'(0)) [2\pi] \equiv \arg_{x-D}(x - \gamma(t)) - \text{Arg}(-\gamma'(0)) [2\pi].$$

On the other hand, all the terms are continuous and both terms agree as  $t$  goes to 0 (by our definition of  $\arg_{x-D}$ ) so we are done.  $\square$

As a corollary of the eq. (2.8) and Lemma 2.3 we have the following formula for all smooth self avoiding curves in a domain  $D$ ,

$$W_{\text{int}}(\gamma) - W(\gamma, \gamma(t)) = \arg_{\gamma(0)-D}(\gamma(0) - \gamma(t)) - \text{Arg}(-\gamma'(0)) \quad (2.8)$$

where  $\arg_{\gamma(0)-D}$  is defined as in Lemma 2.3.

## 2.4 Wilson's algorithm

Given a finite weighted, directed irreducible graph  $G$ , recall that a spanning tree is a spanning subgraph with no (undirected) cycle oriented towards a fixed vertex called the root. The measure defined in eq. (1.2) is called (with a small abuse of language) the Uniform Spanning Tree measure or UST for short.

A crucial tool for studying the UST is Wilson's celebrated algorithm, which we recall here for convenience (since we need it in the context of directed and weighted graphs). In the first step, we fix a vertex, the root. To sample a uniform spanning tree rooted at that vertex, we now pick any vertex and perform a loop erased random walk: that is, we chronologically erase the loops in a simple random walk starting from that vertex until it hits the root. We now define the updated root by adding this loop erased path to the old root and iterate this procedure until there are no isolated vertices remaining. It is easy to see that such a process in the end produces a tree. This tree has the law of the uniform spanning tree ([33]). Further, the law of this tree does not depend on the choice of the ordering of vertices from which we draw loop erased paths while performing Wilson's algorithm.

It follows that from every vertex, one can generate branches of the uniform spanning tree from that vertex to the root by sampling loop erased random walk from those vertices to the root. Often we will be interested in the **wired** uniform spanning tree, which is the UST on the graph where we have identified together a given set of "boundary" vertices to be the root.

## 2.5 Continuum uniform spanning tree and coupling with GFF

Our interest in this section lie in the scaling limits of uniform spanning trees of a domain in the square lattice  $\mathbb{Z}^2$ . In this section, we summarise some results which we shall need later regarding the uniform spanning tree and its coupling with Gaussian free field with certain boundary conditions. More specifically, we need a continuum version of Temperleyan bijection between dimers and spanning trees [26] as is stated in Theorem 2.6. Some of this is implicit in [9, 35]. We will give below a precise statement for completeness below.

**Continuum spanning trees.** The Continuum (Wired) Uniform Spanning Tree, in a simply connected domain  $D$ , is the scaling limit of the above discrete UST in an approximation  $D^{\#\delta}$  of  $D$  where we have wired all vertices on  $\partial D^{\#\delta}$ . The convergence is in the following underlying topology.

Let  $\mathcal{P}(z, w, D)$  be the space of all continuous paths in  $\bar{D}$  from a point  $z \in \bar{D}$  to  $w \in \bar{D}$ . We consider the space  $D \times D \times \cup_{z, w \in \bar{D}} \mathcal{P}(z, w)$ . For any metric space  $X$ , let  $\mathcal{H}(X)$  denote the space of subsets of  $X$  equipped with the Hausdorff metric. We view this space as a subset of the compact space  $\mathcal{H}(\bar{D} \times \bar{D} \times \mathcal{H}(\bar{D}))$  equipped with its metric. This is the **Schramm topology** and we call this space the Schramm space.

All we will need to recall for now is the following fact which is easy to understand. Let  $z_1, z_2, \dots, z_k$  be distinct points in  $D$ . A landmark result of Lawler, Schramm and Werner [31] shows that loop erased random walk in a domain  $D^{\# \delta}$  of  $\mathbb{Z}^2$  approximating  $D$  from a vertex closest to  $z_i$  converges as  $\delta \rightarrow 0$  to a radial SLE<sub>2</sub> curve from a point on the boundary picked according to harmonic measure from  $z_i$ , to  $z_i$  (the convergence is in the Hausdorff sense for curves up to reparametrisation). It is shown in Theorem 1.5 of [39] that the branch from a point in a domain is almost surely unique; it is easy to see that this branch is unique a.s. for Lebesgue-almost every point  $z \in D$ . (Actually Theorem 1.5 in [39] deals with uniform spanning tree in the sphere but the same argument works in a domain, see [39], Theorem 11.1.) It is therefore easy to deduce:

**Proposition 2.4** (Wilson's algorithm in the continuum). *Let  $D$  be a domain and  $z_1, \dots, z_k \in D$ . We can sample the (a.s. unique) branches of the continuum wired UST in a domain  $D$  from  $z_1, \dots, z_k$  as follows. Given the branches  $\eta_i$  from  $z_i$  for  $1 \leq i < j$ , we inductively sample the branch from  $z_j$  as follows. We pick a point  $p$  from the boundary of  $D' = D \setminus \cup_{1 \leq i \leq j} \eta_i$  according to harmonic measure from  $z_j$  and draw an SLE<sub>2</sub> curve in  $D'$  from  $p$  to  $z_j$ . The joint law of the branches do not depend on the order in which we sample the branches.*

**Coupling with a GFF.** Let  $D$  be a domain whose boundary  $\beta$  is a smooth closed curve and let  $x$  be a marked point in the boundary of the domain. Let us parametrise the boundary  $\beta$  of  $D$  in an anticlockwise direction (meaning that  $D$  lies to left of the curve) and such that  $\beta(0) = x$ . We define **intrinsic winding boundary condition** on  $(D, x)$  to be a function  $u$  defined on the boundary by  $u_{(D, x)}(\beta(t)) := W_{\text{int}}(\beta[0, t])$ . We call  $u_{(D, x)}$  the intrinsic winding boundary function and extend it harmonically to  $D$ .

We extend this definition to any domain  $D$  smooth in a neighbourhood of a marked point  $x$  (but not necessarily smooth elsewhere on the boundary and possibly unbounded) as follows. Let  $\varphi : \mathbb{D} \rightarrow D$  be a conformal map which maps  $x$  to 1. Let  $u_{(\mathbb{D}, 1)}$  be the intrinsic winding boundary function on  $(\mathbb{D}, 1)$ . Define  $u_{(D, x)}$  on  $D$  by

$$u_{(\mathbb{D}, 1)} = u_{(D, x)} \circ \varphi - \arg_{\varphi'(\mathbb{D})} \varphi'. \quad (2.9)$$

where we define  $\arg_{\varphi'(\mathbb{D})}$  as the argument defined continuously in  $\varphi'(\mathbb{D})$  (note  $\varphi'(\mathbb{D})$  do not contain 0 since  $\varphi$  is conformal) with the global constant chosen such that  $u_{(D, x)}$  jumps from  $2\pi$  to 0 at  $x$ .

**Remark 2.5.** If the domain  $D$  has a general boundary which might be rough everywhere, then still the definition (2.9) makes sense up to a global additive constant in  $\mathbb{R}$ . In this case there is no way in general to fix this constant since we cannot extend  $\arg \varphi'$  continuously to the boundary in general. We define  $u_{(D, x)}$  in this way for domains with general boundary.

It is elementary but tedious to check that this definition makes sense in the sense that it does not in fact depend on the choice of the conformal map  $\varphi$ : indeed, if one applies a Möbius transform of the disc, winding boundary conditions are changed into winding boundary conditions.

Fix  $\chi = 1/\sqrt{2}$ . This is a notation for an important parameter in the theory of imaginary geometry [34, 35] whose value for  $\kappa = 2$  is  $1/\sqrt{2}$ .



**Theorem 2.6** (Imaginary geometry coupling). *Let  $D$  be a domain with a marked point  $x$  on the boundary. Let  $h = \chi u_{(D,x)} + h_{\text{GFF}}^0$  where  $h_{\text{GFF}}^0$  is a GFF with Dirichlet boundary conditions (where  $u_{(D,x)}$  is defined as in (2.9) and Remark 2.5). There exists a coupling between the continuum wired UST on  $D$  and  $h$  such that the following is true. Let  $\{\gamma_i\}_{1 \leq i \leq k}$  be the branches of the continuum wired UST from points  $\{z_i\}_{1 \leq i \leq k}$  in  $D$ . Then the conditional law of  $h$  given  $\{\gamma_i\}_{1 \leq i \leq k}$  is the same as  $\chi u_{(D \setminus \cup_{1 \leq i \leq k} \gamma_i, x)} + h_{\text{GFF}}^0$ . Furthermore, the GFF is completely determined by the UST and vice-versa.*

*Proof Sketch.* This theorem is essentially a corollary of the results in [31, 34, 35, 9]. The proof uses the theory of imaginary geometry developed in last three of these papers where one realises the SLE curves as *flow lines* of an underlying GFF in the same probability space. We refer to these papers for details of this theory and we provide a brief sketch.

Let  $\mathcal{T}^{\# \delta}$  denote the wired UST of  $\delta \mathbb{Z}^2 \cap D$  which converges to the continuum  $\mathcal{T}$  in the Schramm topology using Lawler, Schramm and Werner [31]. It is also known that the interface between the  $\mathcal{T}^{\# \delta}$  and its dual converges to a continuum space-filling loop  $\eta'$  in *peano curve topology* (see [1], Theorem 4.1). It is known using [34], Theorem 1.1 that this continuum space filling loop  $\eta'$  can be realised as a *counterflow line* of a GFF with boundary condition given by  $u_{D,x}$ <sup>2</sup>. Furthermore, a form of a SLE duality result was also proved in Theorem 1.8 in [35]: suppose we stop  $\eta'$  when it hits the points  $z_1, z_2, \dots, z_k$ . Then the left boundary from  $z_i$  at that time is given by the flow line  $\eta_i$  with angle  $\pi/2$  of the same underlying GFF. Being a flow line, the GFF in the complement of  $(\eta_1, \dots, \eta_k)$  has the right property we require. Using the fact that the peano curve topology is stronger than the Schramm topology, one can see that the  $\eta_i$  are the scaling limit of the branches of the discrete wired UST from a point close to  $z_i$ . This identifies the branches of the wired UST with the flow lines which completes the proof.  $\square$

### 3 Continuum windings and GFF

#### 3.1 Winding in the continuum and statement of the result

The goal of this section is to show that the winding of the branches in a continuum UST gives a Gaussian free field. By analogy with the discrete, we wish to show that the *intrinsic winding* (in the sense of earlier definitions) of the branches of the continuum UST up to the end points is the Gaussian free field. However there are two obstacles if we want to deal with this. Firstly, the branches are rough and hence intrinsic winding does not make sense. Secondly, the winding up to the end point blows up because the branches wind infinitely often in every neighbourhood of their endpoints (indeed this should be the case since the GFF is not defined pointwise).

To tackle the first problem, we note that the topological winding is well defined even for rough curves. We will therefore study the topological winding and add the correction term from eq. (2.8) by hand (see also the remark immediately after eq. (2.8) as well as Remark 3.8).

We address the second problem by regularizing the winding to obtain a well defined function. The regularisation we use is simply to truncate the UST branches at some point. We will therefore have to show that this regularised winding field converges to a GFF as the regularisation vanishes.

Let  $D$  be a bounded domain with a locally connected boundary and a marked point  $x$  on its boundary. Let  $\mathcal{T}$  be a continuum wired uniform spanning tree in  $D$ . Recall that viewed as a

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<sup>2</sup> This very roughly means that one can find a coupling between  $\eta'$  and a GFF so that  $\eta'$  can be thought of as a “flow line” of the GFF

random variable in Schramm's space, a.s. for Lebesgue-almost every  $z \in D$  there is a unique branch connecting  $z$  to  $\partial D$  and for a fixed  $z$  this has the law of a radial SLE<sub>2</sub> (more precise description follows). For  $z \in \mathbb{D}$ , let  $\gamma_z$  be the UST branch to the boundary (let  $z^*$  be the point where it hits  $\partial D$ ) then continues clockwise along  $\partial D$  from  $z^*$  to  $x$ . Note that since  $\partial U$  is locally connected, we can think of  $\partial U$  as a curve with some parametrisation [37] and hence this description indeed makes sense. Recall that for any point  $z$ ,  $z^*$  has the distribution of a sample from the harmonic measure on the boundary seen from  $z$ , which we denote by  $\text{Harm}_D(z, \cdot)$ . Also given  $z^*$ , the part of the curve between  $z^*$  to  $z$  is a radial SLE<sub>2</sub> curve in  $D$  from  $z^*$  to  $z$  in law.

Let us assume for a minute that the boundary of  $D$  is analytic in a neighbourhood of  $x$ . Let us parametrise the part of  $\gamma_z$  for any  $z \in D$  which lies in  $\partial D$  by  $[-1, 0]$  (with some choice of parametrisation) so that  $\gamma_z(-1) = x$  and  $\gamma_z(0) = z^*$ . For  $t \geq 0$ , define  $\gamma_z[0, t]$  the UST branch from  $z^*$  to  $z$  up to to capacity  $t - \log(R(z, D))$  (recall that for every  $z$ , a.s. this is a radial SLE<sub>2</sub> curve targeted at  $z$ ). Define

$$h_t^D(z) = W(\gamma_z[-1, t], z) - \text{Arg}(-\gamma'(-1)) + \arg_{x-D}(x - z) \quad (3.1)$$

where  $\arg_{x-D}$  is defined as in eq. (2.8), i.e., it matches with  $\text{Arg}$  in the direction  $-\gamma'(-1)$ .

The intuition behind adding these extra terms is to work with (an emulation of) the intrinsic winding rather than the topological one via the formula derived in eq. (2.8). Note that  $h_t$  is defined almost surely as an almost everywhere function and hence in particular can be viewed (a.s.) as a random distribution.

For a domain  $D$  with general rough boundary, the additive constant becomes ambiguous. This motivates us to define the following. Let  $\arg_{x-D}(x - z)$  be the argument defined in the domain  $x - D$  up to a global constant in  $\mathbb{R}$ . Now define

$$h_t^D(z) = W(\gamma_z[-1, t], z) + \arg_{x-D}(x - z) \text{ up to a global constant in } \mathbb{R} \quad (3.2)$$

To lighten notation, we shall write  $h_t$  in place of  $h_t^D$  where there is no chance of confusion.

The winding of a single SLE branch has been studied extensively starting with the original paper of Schramm [39] itself. In particular, Schramm obtained the following result:

**Theorem 3.1** ([39], Theorem 7.2). *Suppose  $D = \mathbb{D}$  where  $\mathbb{D}$  is the unit disc. We have the following equality in law*

$$W(\gamma_0[0, t], 0) = B(2t) + y_t, \quad W(\gamma_0[-1, 0], 0) = \Theta$$

where  $B(\cdot)$  is a standard Brownian motion started from 0,  $y_t$  is a random variable having uniform exponential tail and  $\Theta \sim \text{Unif}[0, 2\pi)$ . In fact  $e^{i(B(2t)+\Theta)}$  is the driving function for the SLE curve  $\gamma_0$ . Also for all  $s > 0$

$$\mathbb{P}(|\gamma_0(t)| > e^{-t+s}) \leq ce^{-c's} \quad (3.3)$$

for some  $c, c' > 0$ .

As a side note, we remark that it is precisely this observation which led Schramm to conjecture that loop-erased random walk converges to SLE<sub>2</sub>, by combining this result together with Kenyon's work on the dimer model and his computation of the asymptotic pointwise variance.

Coming back to the continuum uniform spanning tree, for a.e.  $z$  we get a branch  $\gamma_z$  which is an SLE<sub>2</sub> and to which Theorem 3.1 naturally applies. In particular, for a.e.  $z$  we get a driving

Brownian motion  $B_z(t)$ , which forms a Gaussian stochastic process when indexed by  $\mathbb{D}$ . Essentially, the next result, which is the main result of this section, says that this Gaussian process converges to the Gaussian free field as  $t \rightarrow \infty$ . (In fact, the result below even deals with the error term  $y_t$ ). Set  $\chi = 1/\sqrt{2}$ , and recall that  $u_{(D,x)}$  is the function which gives the intrinsic winding of the boundary curve  $\partial D$ , harmonically extended to  $D$  (see (2.9)).

**Theorem 3.2.** *Let  $D$  be a bounded domain with locally connected boundary and a marked point  $x \in \partial D$ . Set  $\chi = 1/\sqrt{2}$ . As  $t \rightarrow \infty$ , we have the following convergence in probability:*

$$h_t \xrightarrow[t \rightarrow \infty]{} h_{\text{GFF}}.$$

*The convergence is in the Sobolev space  $H^{-1-\eta}$  for all  $\eta > 0$ , and holds almost surely along the set of integers. Moreover,  $\mathbb{E}(\|h_t - h_{\text{GFF}}\|_{H^{-1-\eta}}^k) \rightarrow 0$  for any  $k \geq 1$ . The limit  $h_{\text{GFF}}$  is a Gaussian free field with variance  $1/\chi$  and winding boundary conditions: i.e., we have*

$$h_{\text{GFF}} = \frac{1}{\chi} h_{\text{GFF}}^0 + \pi/2 + u_{(D,x)}$$

*where  $h_{\text{GFF}}^0$  is a GFF with Dirichlet boundary conditions on  $D$  and  $u_{(D,x)}$  is defined as in eq. (2.9) and Remark 2.5. When the boundary is rough everywhere, the above convergence should be viewed up to a global constant in  $\mathbb{R}$ .*

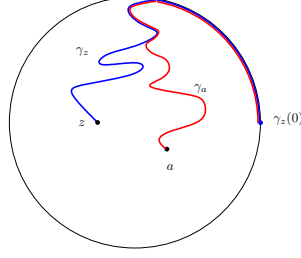
In fact, the coupling between  $(\mathcal{T}, h_{\text{GFF}})$  defined above has the same distribution as the imaginary geometry coupling of Theorem 2.6. In particular, we recover the fact that  $h$  is measurable with respect to  $\mathcal{T}$  (which was already a consequence of [9]). In fact we prove a joint convergence to this pair in Theorem 6.1. The explicit construction of the GFF in terms of  $\mathcal{T}$  which comes from the above result is in our opinion quite simple and in particular has the advantage that it is suited for taking scaling limits, as will be apparent in the proof of the main theorem.

The rest of this section is dedicated to the proof of Theorem 3.2. The general strategy is to first study the  $k$ -point function  $\mathbb{E}[\prod h_t(x_i)]$  and to only integrate them at the last step to obtain moments of the integral of  $h_t$  against test functions. The advantage of working with the  $k$ -point function is that it only depends on  $k$  branches of the tree, which we know how to sample using Proposition 2.4. The first step is therefore to obtain a precise understanding of the behaviour of the winding under conformal maps, which is done in Section 3.2. Given that, the existence of  $\lim_{t \rightarrow \infty} \mathbb{E}[\prod h_t(x_i)]$  follows from relatively simple distortion arguments and is proved in Lemma 3.15. This essentially shows that  $\lim h_t$  exists in the sense of moments. To identify the limit, we again use the behaviour of the winding under conformal maps to show that the conditional expectation of  $\lim h_t$  given some tree branches agrees with the imaginary geometry definition (Sections 3.4 and 3.5). The uniqueness in imaginary geometry concludes. Finally, Section 3.6 covers the extension from the disc to general smooth domains, which is essentially an application of Section 3.2, and Section 3.7 upgrades the convergence from finite dimensional marginals to  $H^{-1-\eta}$  using the moment bounds derived earlier.

Let us note that it would be interesting to be able to compute  $\lim_{t \rightarrow \infty} \mathbb{E}[\prod h_t(x_i)]$  directly as it would provide an alternate self-contained construction of the imaginary geometry coupling.

### 3.2 Change in intrinsic winding via conformal map

The goal of this first section is to understand how the winding of a curve changes when a conformal map is applied to it. The final outcome is Corollary 3.7 which is one of the key deterministic



**Figure 6:** A typical application of Lemma 3.4. Here  $\gamma = \gamma_z$  (the curve going from 1 to  $z$  in a continuum UST) and  $\psi$  is the Loewner map removing  $\gamma_a$ . Note that  $\psi(\gamma_z)$  is well-defined as a continuous curve in  $\mathbb{D}$ . so condition (iii) is satisfied.

statements used in this paper: it states that the change in winding under an application of conformal map  $f$  is roughly  $\arg f'$ . Before stating the lemma we first recall some known estimates from conformal maps.

**Lemma 3.3** (Distortion estimates). *Let  $D$  be a domain containing 0 and let  $\delta = R(0, D)$ . Let  $g$  be a conformal map defined on  $D$  mapping 0 to 0. Then for any  $z \in B(0, \delta/8)$ ,*

$$|g(z)| \leq 4|zg'(0)| \quad ; \quad |g(z) - g'(0)z| < 6\frac{|z|^2}{\delta}|g'(0)|.$$

*In particular, the image of a straight line joining a point at a distance  $\varepsilon < \delta/8$  to 0 under  $g$  lies within a cone of angle  $\arctan(6\frac{\varepsilon}{\delta})$ .*

*Proof.* The first statement follows from applying the Growth theorem (see Theorem 3.23 in [30]) to the function  $g(\delta z/4)/(\delta g'(0)/4)$ , which is defined on the unit disc by Koebe's 1/4 theorem. The second statement follows from applying Proposition 3.26 in [30] with  $r = 1/2$  to the same function.  $\square$

Now we state and prove the lemma about the behaviour of winding under conformal map.

**Lemma 3.4.** *Let  $D, D'$  be domains with locally connected boundary. Let  $\psi$  be conformal map sending  $D$  to  $D'$ . Let  $\gamma : [0, 1] \mapsto \bar{D}$  be a curve (not necessarily self avoiding) in  $\bar{D}$ . Assume the following*

- (i) *The endpoints of  $\gamma$  are smooth and simple.*
- (ii)  *$\arg \psi'$  extends continuously to a neighbourhood in  $D$  of  $\gamma(0)$  and  $\gamma(1)$ .*
- (iii) *There exists a continuous curve  $\tilde{\gamma} \subset \bar{\mathbb{D}}$  such that  $\varphi(\tilde{\gamma}) = \gamma$ , where  $\varphi : \bar{\mathbb{D}} \mapsto \bar{D}$  is the extension of a conformal map in  $\mathbb{D}$ . (Such an extension exists by our assumption that the boundary is locally connected. See Remark 3.5 below for an equivalent reformulation of this condition.)*

*Then,*

$$W(\psi(\gamma), \psi(\gamma(1))) + W(\psi(\gamma), \psi(\gamma(0))) = W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) \\ + \arg_{\psi'(D)}(\psi'(\gamma(1))) - \arg_{\psi'(D)}(\psi'(\gamma(0)))$$

*where  $\arg_{\psi'(D)}$  here is any determination of the argument on the image of  $\psi'$ .*

*Proof.* First, we assume that  $D$  is the unit disc  $\mathbb{D}$  and  $\tilde{\gamma} = \gamma$ . We define a sequence of maps,

$$\psi_t(z) = \psi(tz)/t \text{ for } t \in (0, 1], \quad \psi_0(z) = \psi'(0)z.$$

The map  $\psi_0$  is just a rotation and scaling so the topological winding does not change and  $\arg(\psi'_0(\gamma(1))) = \arg(\psi'_0(\gamma(0))) = \arg(\psi'(0))$ . Thus we have:

$$W(\psi_0(\gamma), \psi_0(\gamma(1))) + W(\psi_0(\gamma), \psi_0(\gamma(0))) = W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) + \arg(\psi'_0(\gamma(1))) - \arg(\psi'_0(\gamma(0))).$$

On the other hand, since locally at  $\gamma(0)$  and  $\gamma(1)$ ,  $\psi$  acts only by rotation and scaling we see that for all  $t$ ,

$$W(\psi_t(\gamma), \psi_t(\gamma(1))) + W(\psi_t(\gamma), \psi_t(\gamma(0))) \equiv W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) + \arg(\psi'_t(\gamma(1))) - \arg(\psi'_t(\gamma(0))) [2\pi]. \quad (3.4)$$

Indeed, note that

$$\begin{aligned} W(\psi_t(\gamma), \psi_t(\gamma(1))) &\equiv \lim_{s \rightarrow 1} \text{Arg}(\psi_t(\gamma(s)) - \psi_t(\gamma(1))) - \text{Arg}(\psi_t(\gamma(0)) - \psi_t(\gamma(1))) [2\pi] \\ &\equiv \text{Arg}(\psi'_t(\gamma(1))) + \text{Arg}(\gamma'(1)) + \pi - \text{Arg}(\psi_t(\gamma(0)) - \psi_t(\gamma(1))) [2\pi] \end{aligned}$$

and

$$\begin{aligned} W(\psi_t(\gamma), \psi_t(\gamma(0))) &\equiv \text{Arg}(\psi_t(\gamma(1)) - \psi_t(\gamma(0))) - \lim_{s \rightarrow 0} \text{Arg}(\psi_t(\gamma(s)) - \psi_t(\gamma(0))) [2\pi] \\ &\equiv \text{Arg}(\psi_t(\gamma(1)) - \psi_t(\gamma(0))) - \text{Arg}(\psi'_t(\gamma(0))) - \text{Arg}(\gamma'(0)) [2\pi] \end{aligned}$$

and also

$$W(\gamma, \gamma(1)) + W(\gamma, \gamma(0)) \equiv \text{Arg}(\gamma'(1)) - \text{Arg}(\gamma'(0)) [2\pi].$$

Both sides of eq. (3.4) match when  $t = 0$  and the right hand side is clearly continuous in  $t$  since  $\psi'$  extends continuously to  $\gamma(0)$  and  $\gamma(1)$ , so we only have to argue that the left hand side is continuous. First, note that the curves  $\psi_t(\gamma)$  are continuous in  $t$  in the Hausdorff sense. Indeed the continuity is obvious for  $t < 1$  and at  $t = 1$  continuity follows from the fact that the map  $\psi$  extends continuously to the boundary of  $\mathbb{D}$ . Now we argue that the topological winding of the curves are continuous in  $t$ , which will follow from the continuity of  $\arg(\psi')$  up to the boundary. Let us argue for the winding around  $\gamma(0)$  at a fixed time  $t_0$ . Since  $\gamma$  is smooth in a neighbourhood of  $\gamma(0)$ , we can find  $\varepsilon > 0$  such that  $|\arg \gamma'(s) - \arg \gamma'(0)| \leq \varepsilon$  for all  $s \leq \varepsilon$ . We can further assume by continuity of  $\arg \psi'$  that  $\varepsilon$  is such that  $|\arg(\psi'_t(\gamma(s)) - \arg(\psi'_{t_0}(\gamma(0)))| \leq \varepsilon$  for  $s \leq \varepsilon$  and  $|t - t_0| \leq \varepsilon$ . This shows that  $\psi_t(\gamma[0, \varepsilon])$  is a smooth curve whose tangent at any point is always within  $2\varepsilon$  of  $\arg(\psi'_{t_0}(\gamma(0)) + \arg \gamma'(0))$ . It is easy to see that such a curve cannot exit a cone of direction  $\arg(\psi'_{t_0}(\gamma(0)) + \arg \gamma'(0))$  and of angle  $2\varepsilon$  and therefore satisfies  $W(\psi_t(\gamma[0, \varepsilon]), \psi_t(\gamma(0))) \leq 2\varepsilon$ . On the other hand  $\psi_t(\gamma[\varepsilon, 1])$  stays uniformly away from  $\psi_t(\gamma(0))$  so  $W(\psi_t(\gamma[\varepsilon, 1]), \psi_t(\gamma(0)))$  is continuous in  $t$ . Overall  $W(\psi_t(\gamma), \psi_t(\gamma(0)))$  is continuous in  $t$ . The argument for the winding around  $\gamma(1)$  is identical and we are done in the case  $D = \mathbb{D}$ .

For the general case, take the conformal map  $\psi \circ \varphi : \mathbb{D} \mapsto D'$ . Using our previous argument for  $\varphi : \mathbb{D} \mapsto D$  and  $\psi \circ \varphi$  gives two equations connecting the winding of  $\tilde{\gamma}$  with the winding of  $\gamma$  and  $\psi(\gamma)$ . Combining the two and noting that the equation do not depend on the particular realisation of  $\arg$ , we conclude.  $\square$

**Remark 3.5.** The condition (iii) in Lemma 3.4 is easier to understand when appealing to the notion of *conformal boundary* (see [6]). To explain what that is, we fix a conformal map  $\psi : D \rightarrow \mathbb{D}$  and equip  $D$  with the distance induced by  $\psi$ , i.e., we set for  $z, z' \in D$ ,  $d_\psi(z, z') = |\psi(z) - \psi(z')|$ . The conformal closure  $\text{cl}(D)$  of  $D$  is defined as the completion of  $D$  with respect to this distance. (The conformal boundary can then be defined to be  $\text{cl}(D) \setminus D$ ; this notion is then equivalent to that of Poisson and Martin boundaries induced by Brownian motion on  $D$ , as well as prime ends in [37].) Note that  $\text{cl}(D)$  is then identified with the closed disc  $\bar{\mathbb{D}}$  and so a point in  $\text{cl}(D)$  projects to a unique point on  $\bar{D}$  since by local connectedness of  $\partial D$ ,  $\psi^{-1}$  extends to the closed disc. (However the converse is not true in general if the domain is not locally connected, see Theorem 2.6 in [37]).

With these definitions, assumption (iii) simply says that  $\gamma$  is a continuous curve in  $\text{cl}(D)$ , i.e.,  $\gamma$  is the projection of a continuous curve in  $\text{cl}(D)$ .

Putting Lemma 3.4 together with Lemma 2.3, we obtain the following.

**Lemma 3.6.** *Let  $D, D'$  be domains with locally connected boundary and let  $\psi$  be conformal map sending  $D$  to  $D'$ . Let  $\gamma : [0, 1] \mapsto \bar{D}$  be a curve in  $\bar{D}$  and assume that it is smooth and simple at  $\gamma(1)$ . Assume further that  $\arg(\psi')$  extends continuously to  $\gamma(0)$  and  $\gamma(1)$ . Then,*

$$W(\psi(\gamma), \psi(\gamma(1))) - W(\gamma, \gamma(1)) = \arg_{\psi'(D)}(\psi'(\gamma(1))) + \arg_{\gamma(0)-D}(\gamma(0) - \gamma(1)) \\ - \arg_{\psi(\gamma(0))-D'}(\psi(\gamma(0)) - \psi(\gamma(1))).$$

where  $\arg_{\gamma(0)-D}$  and  $\arg_{\psi(\gamma(0))-D'}$  are defined as in Lemma 2.3 and  $\arg_{\psi'(D)}$  is defined so that,

$$\lim_{z \rightarrow \gamma(0), z \in D} \left( \arg_{\psi(\gamma(0))-D'}(\psi(\gamma(0)) - \psi(z)) - \arg_{\gamma(0)-D}(\gamma(0) - z) \right) = \text{Arg}(\psi'(\gamma(0))) = \arg_{\psi'(D)}(\psi'(\gamma(0))) \quad (3.5)$$

Furthermore if  $\arg(\psi')$  does not extend continuously to  $\gamma(0)$ , the formula still holds up to a global constant in  $\mathbb{R}$  not depending on  $\gamma$  and depending only on the choice of the constants in for the arguments.

*Proof.* First assume that  $\gamma(0)$  is a smooth simple point of  $\gamma$ , then the results follows directly from replacing Lemma 3.4 together with Lemma 2.3. Now to get rid of this assumption, note that if we modify  $\gamma$  in a small enough neighbourhood of its starting point, we cannot change the value of the left hand side. We can therefore always change  $\gamma$  into a curve satisfying our assumptions so we are done. For the case where  $\arg(\psi')$  does not extend to  $\gamma(0)$ , note that as in the proof of Lemma 3.4, we can approximate  $\psi$  by a map  $\psi_t$  that has derivative in  $\gamma(0)$ . Applying the result for the smooth case modulo a global constant and then taking the approximation to 0 (i.e.  $t \rightarrow 1$ ) yields the result.  $\square$

Finally in the case where we want to compute winding with respect to a point different from the endpoint, the distortion lemma gives us a version of the corollary with an error term.

**Corollary 3.7.** *Let  $D, D'$  be bounded domains with locally connected boundary and let  $\psi$  be conformal map sending  $D$  to  $D'$ . Let  $\gamma : [0, 1] \mapsto \bar{D}$  be a curve in  $\bar{D}$ . Assume further that  $\arg(\psi')$  extends continuously to  $\gamma(0)$  and  $\gamma(1)$ . Let  $z$  be a point in  $D$  and let  $\delta = R(z, D)$  be its conformal radius and assume that  $|z - \gamma(1)| \leq \delta/8$ . Then,*

$$W(\psi(\gamma), \psi(z)) - W(\gamma, z) = \arg_{\psi'(D)}(\psi'(z)) + \arg_{\gamma(0)-D}(\gamma(0) - z) \\ - \arg_{\psi(\gamma(0))-D'}(\psi(\gamma(0)) - \psi(z)) + O(|z - \gamma(1)|/\delta), \quad (3.6)$$



where the implicit constant in the  $O(|z - \gamma(1)|/\delta)$  is universal. The constants in the arguments are defined as in Lemma 3.6. Furthermore if  $\arg(\psi')$  does not extend to  $\gamma(0)$ , the formula still holds up to a global constant in  $\mathbb{R}$  depending on the choice of the constants for the arguments and not on  $\gamma$ .

*Proof.* Let  $\tilde{\gamma}$  be obtained by appending a straight line connecting  $\gamma(1)$  to  $z$ . By Koebe's 1/4 theorem  $\tilde{\gamma}$  is still a curve in  $D$  and it is obviously smooth at its last point so we can apply Lemma 3.6 to it. On the other hand, by Lemma 3.3, we see that the image of the straight segment has winding  $O(|z - \gamma(t)|/\delta)$  since it stays in a cone of that angle. By additivity of the winding we are done.  $\square$

**Remark 3.8.** In the SLE/GFF coupling results developed by Dubédat, Miller and Sheffield [9, 35] (referred to as imaginary geometry), the harmonic extension of intrinsic winding of an SLE curve was defined using a change in coordinate formula under conformal map. Corollary 3.7 is essentially a direct proof of this change of coordinate formula for topological winding. This along with eq. (2.8) shows that the intrinsic notion of winding is consistent with the formulation in [35, 9]: that is, the boundary conditions of the free field in imaginary geometry can be viewed as the harmonic extension of the intrinsic winding (while the intrinsic winding itself doesn't make sense, its harmonic extension does).

Note that the condition (ii) in Lemma 3.4 is trivial if  $\gamma(0)$  and  $\gamma(1)$  are interior points but is non-trivial for boundary points. However Theorem 3.2 in [37] shows that if  $D, D'$  are smooth Jordan curves then (ii) is satisfied. We now give a slight generalisation of that theorem which gives us a simple geometric sufficient condition for (ii).

**Lemma 3.9.** *Let  $\psi$  be a conformal map between two domains  $D$  and  $\tilde{D}$  and let  $x \in \partial D$  be fixed. Assume that  $D$  and  $\tilde{D}$  have locally connected boundary and let  $\lambda$  be a parametrisation of  $\partial D$  coming from a map to the disc (i.e up to parametrisation,  $\lambda$  is the curve  $(g(e^{it}))_{0 \leq t \leq 2\pi}$  with  $g$  conformal from  $\mathbb{D}$  to  $D$ ). If there exists an open interval  $I$  such that  $x \in \lambda(I)$  and both  $\lambda(I)$  and  $\psi(\lambda(I))$  are smooth curves, then  $\arg \psi'$  extends continuously to a neighbourhood of  $x$ . Here  $\arg$  is any realisation of argument in the range of  $\psi'$ .*

*Proof.* We consider first the case  $D = \mathbb{D}$ . Let us write  $y = \psi(x)$  and let  $I$  be an interval given as in the statement. Let  $\tilde{\lambda}$  be a  $C^1$  parametrisation of  $\psi(\lambda(I))$  with non vanishing derivative and  $t_0$  be such that  $y = \tilde{\lambda}(t_0)$ . Since  $\psi(\lambda(I))$  is smooth, we can assume by taking a smaller  $I$  if necessary that for all  $t \in I$ ,  $|\arg \tilde{\lambda}'(t) - \arg \tilde{\lambda}'(t_0)| \leq 1$ . As in the proof of Lemma 3.4, this implies that  $\tilde{\lambda}$  stays in a cone of angle 2 around each of its points and in particular  $\tilde{\lambda}$  is injective, or in other word it is a Jordan arc. It is then easy to see that we can find a sub-domain  $E \subset D'$  such that  $E$  is bounded by a smooth Jordan curve and  $\tilde{\lambda}(t) \in \partial E$  for  $t$  in a neighbourhood of  $t_0$ .

Let  $K = \psi^{-1}(D' \setminus E)$  and let  $g$  be a map sending  $\mathbb{D} \setminus K$  to  $\mathbb{D}$  and  $x$  to  $x$ . Observe that  $\arg(g')$  extends to  $x$  (e.g. by Schwarz reflection). Note also that the map  $\varphi = \psi \circ g^{-1}$  is a conformal map sending  $\mathbb{D}$  to  $E$  which is a smooth Jordan domain. By Theorem 3.2 in [37]  $\arg(\varphi')$  extends continuously to the boundary. Finally by construction  $\psi = \varphi \circ g$  in a neighbourhood of  $x$  so  $\arg \psi'$  also extends to the boundary in a neighbourhood of  $x$ . We conclude with arbitrary  $D$  and  $D'$  by considering the maps from  $\mathbb{D}$  to  $D$  and  $\mathbb{D}$  to  $D'$  and by composition.  $\square$



### 3.3 Convergence in the unit disc

We first prove Theorem 3.2 in the case  $D = \mathbb{D}$  of the unit disc, with the marked point 1. The extension of the results to general domains is discussed in Section 3.6. **Until that section, we henceforth assume  $D = \mathbb{D}$ .**

Recall that the definition of  $h_t$  for this case is given by

$$h_t(z) = W(\gamma_z[-1, t]) + \arg_{1-\mathbb{D}}(1 - z) + \pi/2.$$

As per eq. (2.8),  $\arg_{1-\mathbb{D}}$  is the unique argument in  $1 - \mathbb{D}$  taking values in  $(-\pi/2, \pi/2)$ . The goal of this section is to show that  $h_t$  as a distribution converges in the sense of moments. To do this we first investigate the  $k$  point moment function for which we need the following application of Corollary 3.7.

**Lemma 3.10.** *Let  $a_1, \dots, a_k \in \mathbb{D}$  be distinct and let  $K = \gamma_{a_1}[0, t_1] \cup \dots \cup \gamma_{a_k}[0, t_k]$  where  $0 \leq t_i \leq \infty$ . Fix  $z \in \mathbb{D}$  distinct from any of the  $a_i$ , and  $T > 0$ . Let  $D' = \mathbb{D} \setminus K$  and assume that 1 is a smooth point of  $\partial D'$ . Let  $g : D' \rightarrow \mathbb{D}$  be a conformal map such that  $g(1) = 1$  (note such a map is not unique). Also let  $\delta = R(z, D')$  and suppose  $\gamma_z(T) \in B(z, \varepsilon)$  where  $\varepsilon \leq \delta/8$ . Then*

$$W(g(\gamma_z[-1, T]), g(z)) + \arg_{1-\mathbb{D}}(1 - g(z)) - \arg_{g'(D')}(g'(z)) + \epsilon(T) = W(\gamma_z[-1, T], z) + \arg_{1-\mathbb{D}}(1 - z) \quad (3.7)$$

where  $|\epsilon(T)| = O(\varepsilon/\delta)$  and the implied constant is universal. The arguments are chosen such that  $\arg_{1-\mathbb{D}} \in [-\pi/2, \pi/2]$  and  $\arg_{g'(D')}(1) = 0$ .

Furthermore, assume that  $\text{dist}(z, K \cup \partial \mathbb{D}) > \text{dist}(z, \gamma_z[0, T])$ . Then

$$\frac{e^{-T}}{4} \leq \frac{R(g(z), \mathbb{D} \setminus g(\gamma_z[0, T]))}{R(g(z), \mathbb{D})} \leq e^{-T} \text{dist}(z, K \cup \partial \mathbb{D})^{-1} \quad (3.8)$$

*Proof.* The first part is just an application of Corollary 3.7. Indeed we can apply this corollary since any SLE curve can be seen as a conformal image of an SLE curve in the unit disc and clearly  $\arg \psi'$  extends continuously to 1 using Lemma 3.9 since 1 remains a smooth point on the boundary almost surely. So we only have to check the choice of the constant in the arguments. First note a.s. 1 remains a smooth point and  $\arg_{1-D'}$  and  $\arg_{1-\mathbb{D}}$  match near 0. Thus  $\arg_{g'(D')}(1) = 0$  agrees with the prescription from Corollary 3.7.

We now check the way the capacity of the curve  $\gamma_z = \gamma_z[0, T]$  is changed when applying the map  $g$ . Note that by Schwarz's lemma, if  $D' = D'_T$ ,

$$|g'(z)| = \frac{R(g(z), \mathbb{D})}{R(z, D')} \leq R(g(z), \mathbb{D}) / \text{dist}(z, K \cup \partial \mathbb{D}).$$

Hence by conformal covariance and domain monotonicity of the conformal radius,

$$R(g(z), \mathbb{D} \setminus g(\gamma_z)) = R(z, \mathbb{D} \setminus (K \cup \gamma_z)) |g'(z)| \leq |g'(z)| R(z, \mathbb{D} \setminus \gamma_z) \leq R(g(z), \mathbb{D}) e^{-T} \text{dist}(z, K \cup \partial \mathbb{D})^{-1}$$

which gives the upper bound. In the other direction, we find (since  $|g'(z)|/R(g(z), \mathbb{D}) = 1/R(z, D') \geq 1$  by domain monotonicity of the conformal radius),

$$\frac{R(g(z), \mathbb{D} \setminus g(\gamma_z))}{R(g(z), \mathbb{D})} \geq R(z, \mathbb{D} \setminus \gamma_z \cup K) \geq \text{dist}(z, \gamma_z \cup K \cup \partial \mathbb{D}) \geq e^{-T}/4$$

since by our assumption that  $\gamma_z$  is closer to  $z$  than  $A \cup \partial \mathbb{D}$  and Koebe's 1/4 theorem. This proves (3.8).  $\square$

Theorem 3.1 deals with SLE curves towards 0. We now provide an extension of this result for SLE curves towards an arbitrary point in the unit disc.

**Lemma 3.11.** *Let  $z \in \mathbb{D}$  and let  $\psi : \mathbb{D} \mapsto \mathbb{D}$  be the Möbius transformation mapping  $z$  to 0 and 1 to 1. If the tip of  $\gamma_z[-1, t]$  lies in  $B(z, \varepsilon)$  where  $\varepsilon \leq R(z, \mathbb{D})/8$ , then we have the following pointwise relation:*

$$W(\psi(\gamma_z[-1, t]), 0) = W(\gamma_z[-1, t], z) - \arg_{1-\mathbb{D}}(1-z) + \epsilon(t) \quad (3.9)$$

where the error term  $|\epsilon(t)| \leq c\varepsilon/R(z, \mathbb{D})$  for some universal constant  $c > 0$  and  $\arg_{1-\mathbb{D}}$  is the argument function in the domain  $1-\mathbb{D}$  with value in  $(-\pi/2, \pi/2)$ . Also for all  $s, t$

$$\mathbb{P}(|\gamma_z(t) - z| > e^{-t+s} R(z, \mathbb{D})) \leq ce^{-c's} \quad (3.10)$$

where  $c, c'$  are independent of  $z$ .

*Proof.* We are going to apply Corollary 3.7 to the Möbius transform  $\psi$  of  $\mathbb{D}$  mapping  $z$  to 0 and 1 to 1.

Computing  $\psi(t)$  for  $t \in \mathbb{D}$ :

$$\psi'(t) = \frac{1 - |z|^2}{(1 - t\bar{z})^2} \frac{1 - \bar{z}}{1 - z}. \quad (3.11)$$

Now we argue that  $\arg_{\psi'(\mathbb{D})}(\cdot)$  can be taken to be  $\text{Arg} \in (-\pi, \pi]$  with branch cut  $(-\infty, 0]$ . Indeed let us describe  $\psi'(\mathbb{D})$ . First,  $1 - \bar{z}\mathbb{D}$  is a disc of radius  $|z|$  contained in the right half plane  $\{z : \Re(z) > 0\}$ . Therefore  $1/(1 - \mathbb{D}\bar{z})$  is a subset of the right half plane since  $z \mapsto 1/z$  preserves the right half plane. Thus  $(1 - \mathbb{D}\bar{z})^{-2}$  does not contain  $(-\infty, 0]$ . Multiplying by the positive real number  $(1 - |z|^2)$  and rotating by  $\theta = \text{Arg}((1 - \bar{z})/(1 - z))$ , we see that  $\psi'_1(\mathbb{D})$  does not contain the half line  $w$  joining 0 and  $e^{i(\theta-\pi)}$ . Therefore  $\arg_{\psi'(\mathbb{D})}$  coincides with the argument  $\tilde{\text{Arg}}$  with branch cut  $w$  that can take the value 0 (since they differ by some multiple of  $2\pi$  and agree at  $\psi'(1)$  by definition). With our choice of  $\theta$ ,  $\tilde{\text{Arg}}(\cdot)$  takes values in  $(\theta - \pi, \theta + \pi]$ . Therefore  $\tilde{\text{Arg}}((1 - \bar{z})/(1 - z)) = \text{Arg}((1 - \bar{z})/(1 - z))$  and hence,

$$\begin{aligned} \arg_{\psi'(\mathbb{D})}(1 - \bar{z})/(1 - z) &= \tilde{\text{Arg}}(1 - \bar{z})/(1 - z) = \theta \\ &= \text{Arg}(1 - \bar{z})/(1 - z) = -2 \text{Arg}(1 - z) = -2 \arg_{1-\mathbb{D}}(1 - z) \end{aligned}$$

where in the second line we used that  $1 - z$  is in the right half-plane, so the additivity of the arguments holds. Plugging this in Corollary 3.7 and making the cancellation, we have the desired expression (3.9).

Now without loss of generality, assume  $t - s \geq 10$ . Applying Koebe's 1/4 theorem (see Theorem 3.17 in [30]), we see that  $B(0, e^{-t+s}/4) \subset \psi_1(B(z, e^{-t+s}R(z, \mathbb{D})))$ . Hence applying (3.3) in Theorem 3.1 and conformal invariance we have (3.10).  $\square$

We now want to regularise  $h_t$  a bit further by restricting it to an event where the tip is not too far away from the endpoint. Define for  $t \geq 0$  and  $z \in \mathbb{D}$ ,

$$\mathcal{A}(t, z) := \{|\gamma_z(t) - z| < e^{-t/2} R(z, \mathbb{D})\} \quad ; \quad \hat{h}_t(z) := h_t(z) \mathbb{1}_{\mathcal{A}(t, z)}.$$

By Lemma 3.11,  $\mathcal{A}(t, z)$  is a very likely event:

$$\mathbb{P}(\mathcal{A}(t, z)) > 1 - ce^{-c't} \quad (3.12)$$

for some universal constants  $c, c' > 0$ .

**Lemma 3.12.** *We have for every  $z \in \mathbb{D}$ ,*

$$\lim_{t \rightarrow \infty} \mathbb{E}(\hat{h}_t(z)) = \frac{3\pi}{2} + 2 \arg_{1-\mathbb{D}}(1-z) \quad (3.13)$$

*Also, we have the following bounds on the moments:*

$$\mathbb{E}(|\hat{h}_t(z)|^k) \leq c(k)(1+t^{k/2}) \quad \mathbb{E}(|h_t(z)|^k) \leq c(k)(1+t^{k/2}) \quad (3.14)$$

*Proof.* We first check eq. (3.13) and eq. (3.14) for  $z = 0$ . Then  $\gamma_z(0)$  is uniformly distributed on  $\partial\mathbb{D}$ , contributing an expected topological winding of  $\pi$ . Adding the term  $\pi/2$  in the definition of  $h_t$  in eq. (3.1) shows that  $\mathbb{E}(h_t) = 3\pi/2$  (using the fact that the Loewner equation is invariant under  $z \mapsto \bar{z}$ ). Furthermore, by Schramm's theorem (Theorem 3.1), we have  $\mathbb{E}(h_t(0)^2) \leq 2t + o(t)$  and since  $\mathbb{P}(\mathcal{A}(t, z)) \geq 1 - e^{-ct}$  we deduce from Cauchy–Schwarz that  $\lim_{t \rightarrow \infty} \mathbb{E}(\hat{h}_t(0)) = 3\pi/2$ . The moment bound for  $h_t(0)$  follows again from Theorem 3.1 and the inequality  $|a+b|^k \leq 2^{k-1}(|a|^k + |b|^k)$ .

For any other  $z \in \mathbb{D}$ , we start by proving the moment bound eq. (3.14). We join  $\gamma_z(t)$  and  $z$  by a hyperbolic geodesic in  $\mathbb{D}$ , call the resulting union  $\gamma'$ , and apply  $\psi$  to it. Then the image becomes a concatenation of an  $\text{SLE}_2$  curve targeted towards 0 and another hyperbolic geodesic. Using eq. (3.9) (which is deterministic) with  $\varepsilon = 0$ ,  $W(\gamma', z) - W(\psi(\gamma'), 0) = -\arg_{1-\mathbb{D}}(1-z)$ . Since the winding of the hyperbolic geodesics are bounded by at most  $\pi$ , and the winding of  $\psi(\gamma)$  possesses the required moment bounds, this proves eq. (3.14) in  $\mathbb{D}$ .

Now using (3.9):

$$\mathbb{E}(W(\psi(\gamma_z[-1, t]), 0) \mathbb{1}_{\mathcal{A}(t, z)}) = \mathbb{E}(W(\gamma_z[-1, t], z) \mathbb{1}_{\mathcal{A}(t, z)}) - \arg_{1-\mathbb{D}}(1-z) \mathbb{P}(\mathcal{A}(t, z)) + \mathbb{E}(\epsilon(t) \mathbb{1}_{\mathcal{A}(t, z)}) \quad (3.15)$$

Since  $\psi(\gamma_z[-1, t])$  is an  $\text{SLE}_2$  curve towards 0 the left hand side of (3.15) converges to  $3\pi/2$ . Also from Lemma 3.11, the error term  $|\epsilon(t)| < ce^{-c't} \rightarrow 0$  on  $\mathcal{A}(t, z)$  and hence converges to 0 as  $t \rightarrow \infty$ . Recall also the terms added in the definition of  $h_t$  in (3.1). Combining all these together with eq. (3.14), we have our result.  $\square$

### 3.4 Conformal covariance of $k$ -point function

In the next lemma, we prove the existence of the limit of the  $k$ -point function of the regularised winding field of the continuum UST. However we do not identify the limit at this point as this requires a separate argument. For this separate argument we will also need a convergence result of the  $k$ -point function given several branches of  $\mathcal{T}$ , the continuum UST.

**Proposition 3.13.** *Let  $\{z_1, \dots, z_k, w_1, \dots, w_{k'}\}$  be a set of points in  $\mathbb{D}$  all of which are distinct. Then the following is true.*

- Both  $\lim_{t \rightarrow \infty} \mathbb{E}(\prod_{i=1}^k \hat{h}_t(z_i))$  and  $\lim_{t \rightarrow \infty} \mathbb{E}(\prod_{i=1}^k h_t(z_i))$  exist and are equal. Call this limit  $H(z_1, \dots, z_k)$ .
- Let  $A = \{\gamma_{w_1}, \dots, \gamma_{w_{k'}}\}$  be a set of branches of  $\mathcal{T}$ . Let  $\mathbb{E}^A$  denote the conditional expectation given  $A$ . Let  $g_A : \mathbb{D} \setminus A \mapsto \mathbb{D}$  be a conformal map which fixes 1 (this is non unique). Let  $\tilde{h}_t$  be an independent copy of  $h_t$  in  $\mathbb{D}$ . Then

$$\lim_{t \rightarrow \infty} \mathbb{E}^A\left(\prod_{i=1}^k h_t(z_i)\right) = \lim_{t \rightarrow \infty} \mathbb{E}^A\left(\prod_{i=1}^k \hat{h}_t(z_i)\right) = \lim_{t \rightarrow \infty} \mathbb{E}\left[\prod_{i=1}^k (\tilde{h}_t(g_A(z_i)) - \arg_{g'_A(\mathbb{D} \setminus A)}(g'_A(z_i)))\right] \quad \text{a.s.}$$

We first need the following estimate on hitting probabilities of an  $\text{SLE}_2$ . This is one place where we are using the fact that we are in the regime  $\kappa = 2$ .

**Lemma 3.14.** *Let  $D$  be a domain in  $\mathbb{C}$  with  $z, w \in D$ . Let  $\gamma_w$  be a radial  $\text{SLE}_2$  started from a point on the boundary picked according to harmonic measure from  $w$  and targeted at  $w$ . Let  $\delta = |z - w| \wedge \text{dist}(z, \partial D) \wedge \text{dist}(w, \partial D)$ . Then for all  $0 < \varepsilon < \delta$ , there exists a universal constant  $c_0 > 0$  such that*

$$\mathbb{P}(|\gamma_w - z| < \varepsilon) < \left(\frac{\varepsilon}{\delta}\right)^{c_0}$$

Now suppose  $\varepsilon = \text{dist}(w, \partial D) < \text{Diam}(D)/10$ . Then for  $\varepsilon' > \varepsilon$

$$\mathbb{P}(|\gamma_w \subset B(w, \varepsilon')|) \geq 1 - c \left(\frac{\varepsilon}{\varepsilon'}\right)^{c'}$$

*Proof.* This is a corollary of convergence of loop erased random walk to  $\text{SLE}_2$  in  $\mathbb{Z}^2$  (Lawler, Schramm and Werner [31]) and Proposition 4.8 and Lemma 4.15. Proposition 4.8 is exactly the first statement written for a discrete lattice. For the second statement take  $K = B(w, \varepsilon') \cap \partial D$  in Lemma 4.15.  $\square$

The technical part of the proof of Proposition 3.13 is accomplished in the following lemma.

**Lemma 3.15.** *Let  $\{z_1, z_2, \dots, z_k\}$  be a set of points in  $\mathbb{D}$  all of which are distinct. Let  $r = \min_i \text{dist}(z_i, \partial \mathbb{D}) \wedge \min_{i \neq j} |z_i - z_j|$ . Let  $t_1 \geq t_2 > \dots \geq t_k > t \geq -10 \log r + 1$  such that  $t_1 < 10t_k$ . Then there are constants  $c, c'$  depending only on  $k$  such that*

$$|\mathbb{E}(\prod_{i=1}^k \hat{h}_{t_i}(z_i)) - \mathbb{E}(\prod_{i=1}^k \hat{h}_t(z_i))| < ct^k e^{-c't} \quad , \quad |\mathbb{E}(\prod_{i=1}^k h_{t_i}(z_i)) - \mathbb{E}(\prod_{i=1}^k h_t(z_i))| < ct^k e^{-c't}$$

*Proof.* We first claim that it is enough to prove that for  $t_i$ 's as above,

$$|\mathbb{E}(\prod_i \hat{h}_{t_i}(z_i)) - \mathbb{E}(\prod_i \hat{h}_t(z_i))| \leq ct_1^{k/2} e^{-c't} \quad (3.16)$$

This clearly completes the proof since we can break up the interval  $[t, t_k]$  into  $\cup_{i=1}^J [t2^{i-1}, t2^i]$  where  $t2^{J-1} \leq t_k < t2^J$ . Using the bound (3.16) for each such interval and using  $t_1 < 10t_k$ ,

$$|\mathbb{E}(\prod_i \hat{h}_{t_i}(z_i)) - \mathbb{E}(\prod_i \hat{h}_t(z_i))| \leq c \sum_{i=1}^J (t2^i)^{k/2} e^{-c't2^{i-1}} \leq c \sum_{i=1}^{\infty} (t2^i)^{k/2} e^{-c't2^{i-1}} \leq ct^{k/2} e^{-c't} \quad (3.17)$$

from which Lemma 3.15 follows. The bound for the term involving  $h_t$  follows from that of  $\hat{h}_t$  using eq. (3.14), Holders inequality (generalised for  $k$  terms) and the exponential bound on the probability of events  $\mathcal{A}(t_i, z_i)$ .

To prove (3.16), the idea is to consider several cases depending on how close  $\gamma_{z_i}$  gets to the other points. If it gets very close, the distortion of the conformal map becomes more pronounced and the estimate in Lemma 3.10 carries large errors. But  $\gamma_{z_i}$  getting close to  $z_j$  for some  $j \neq i$  is unlikely by Lemma 3.14 and comes at a price. So there is a tradeoff between these two situations. Let  $\delta_i = \inf_{j \neq i} \text{dist}(z_i, \gamma_{z_j}[0, \infty)) \wedge r$  and  $\delta_{\min} := \min_i (\delta_i)$ .

**Case 1:**  $-\log(\delta_{\min}) > t/4$ . By Lemma 3.14 and a union bound,  $\mathbb{P}(-\log \delta_{\min} > t/4) \leq ck(\frac{e^{-t/4}}{r})^{c_0}$  for universal constants  $c, c_0$ . Using the fact that  $t > -10 \log r + 1$ , we see that  $\mathbb{P}(-\log \delta_{\min} > t/4) \leq cke^{-c't}$  for some  $c' > 0$ . Using the one-point moment bounds (3.14) and Hölder's (generalised) inequality,

$$|\mathbb{E}(\prod_i \hat{h}_{t_i}(z_i)) - \mathbb{E}(\prod_i \hat{h}_t(z_i)) \mathbb{1}_{-\log(\delta_{\min}) > t/4}| < ct_1^{k/2} e^{-c't}. \quad (3.18)$$

for some positive universal constants  $c, c'$  since  $t > 1$ .

**Case 2:**  $-\log(\delta_{\min}) \leq t/4$ . Let  $A_i := \{\gamma_{z_j}[0, \infty) : j \neq i\}$ . First we observe that it is enough to prove

$$|\mathbb{E}(h_{t_i}(z_i) - h_t(z_i) | A_i) \mathbb{1}_{-\log(\delta_{\min}) < t/4}| \leq ct_i e^{-c't} \quad (3.19)$$

since we can use the decomposition

$$|\mathbb{E}(\prod_i \hat{h}_{t_i}(z_i)) - \mathbb{E}(\prod_i \hat{h}_t(z_i))| \leq \sum_{i=1}^k |\mathbb{E}(\hat{h}_{t_i}(z_i) - \hat{h}_t(z_i) | A_i \hat{h}_t(z_1) \dots \hat{h}_t(z_{i-1}) \hat{h}_{t_{i+1}}(z_{i+1}) \dots \hat{h}_{t_n}(z_k))| \quad (3.20)$$

and then use Hölder's inequality, (3.19) and the one-point moment bounds (3.14) to obtain the required bound.

We now concentrate on the proof of (3.19). We wish to use Lemma 3.10 and map out  $A_i$  by a conformal map  $\varphi$  mapping  $z_i$  to 0 and 1 to 1 and record the change in winding of  $\gamma_{z_i}$ . By (3.8),

$$\frac{e^{-t_i}}{4} \leq R(\varphi(z_i), \mathbb{D} \setminus \varphi(\gamma_{z_i}[0, t_i])) \leq e^{-(t_i - t/4)}.$$

since  $-\log(\delta_{\min}) \leq t/4$ . Therefore, using Lemma 3.10 (note that the  $\arg_{\varphi'(\mathbb{D} \setminus A_i)}$  term cancels) for an independent copy  $\tilde{h}$  of  $h$ , we have

$$\mathbb{E}((\hat{h}_{t_i}(z_i) - \hat{h}_t(z_i)) \mathbb{1}_{\mathcal{A}(t, z_i)} | A_{z_i}) = \mathbb{E}\left((\tilde{h}_{t'_i}(z_i) - \tilde{h}_{t'}(z_i) + \epsilon(t_i) - \epsilon(t)) \mathbb{1}_{\mathcal{A}(t, z_i)}\right) \quad (3.21)$$

where  $|t'_i - t_i| < t_i/2$  and  $|t' - t| < t/2$  and  $|\epsilon(t_i)| \vee |\epsilon(t)| \leq e^{-ct}$  on  $\mathcal{A}(t, z_i)$ . Now notice by symmetry,  $\mathbb{E}((\tilde{h}_{t'_i}(z_i) - \tilde{h}_{t'}(z_i))) = 0$ . Now we conclude using Cauchy-Schwarz, the moment bound (3.14) and the bound on the probability on  $\mathcal{A}(t, z_i)^c$ .  $\square$

We also need the following estimate which says that the  $k$ -point function blows up at most like a power of  $\log(r)$  as the points come close.

**Lemma 3.16** (Logarithmic divergence). *For any  $k \geq 1$  and any  $k$  distinct points  $z_1, z_2, \dots, z_k \in \mathbb{D}$  and  $t > 0$ ,*

$$|\mathbb{E}(\prod_{i=1}^k h_t(z_i))| \leq c(1 + \log^k(1/r)) \quad ; \quad |\mathbb{E}(\prod_{i=1}^k \hat{h}_t(z_i))| \leq c(1 + \log^k(1/r))$$

where  $r = \min_i \text{dist}(z_i, \partial\mathbb{D}) \wedge \min_{i \neq j} |z_i - z_j|$  and  $c = c(k) > 0$  is a constant.

*Proof.* We only check the first of these inequalities as the proof of the other is identical. Let  $t = -10 \log r + 1$ . By Lemma 3.12, for  $t' \leq t$ , we obtain  $\mathbb{E}(\prod_{i=1}^k |h_{t'}(z_i)|) \leq C(1 + (t')^{k/2}) \leq C(1 + t^{k/2})$  which is what we wanted for  $t' \leq t$ . On the other hand if  $t' \geq t$ , by Lemma 3.15,  $|\mathbb{E}(\prod_{i=1}^k h_{t'}(z_i)) - \prod_{i=1}^k h_t(z_i)| < ct^k e^{-ct}$ . Combining with the result for  $t' = t$  we obtained the desired bound also for  $t' \geq t$ .  $\square$

*Proof of Proposition 3.13.* Notice that Lemma 3.15 implies that  $\mathbb{E}(\prod_{i=1}^k h_t(z_i))$  (resp.  $\mathbb{E}(\prod_{i=1}^k \hat{h}_t(z_i))$ ) is a Cauchy sequence and hence converges. Moreover,

$$|\mathbb{E}(\prod_{i=1}^k h_t(z_i)) - \prod_{i=1}^k \hat{h}_t(z_i)| \leq \mathbb{E}(\prod_{i=1}^k |h_t(z_i)| \mathbb{1}_{\cup_i \mathcal{A}(t, z_i)^c}) \leq c(1 + t^{k/2})e^{-c't} \rightarrow 0 \quad (3.22)$$

Hence the limits are the same, proving the first point.

To simplify notation, we write  $g$  in place of  $g_A$  and  $\arg$  in place of  $\arg_{g'_A(\mathbb{D} \setminus A)}$ . Let  $r = \min_i \text{dist}(z_i, A \cup \partial \mathbb{D})$ . Take  $t > -11 \log r + 1$ . Notice from Lemma 3.10, we see that (using the obvious domain Markov property and conformal invariance of the UST) given  $A$ ,  $h_t(z)$  has the law of  $\tilde{h}_{t_i}(z) - \arg g'(z_i) + \epsilon_i(t)$ , where

$$t_i = -\log R(g(z_i), \mathbb{D} \setminus g(\gamma_{z_i}[0, t]) + \log R(g(z_i), \mathbb{D}).$$

Hence

$$\mathbb{E}^A(\prod_{i=1}^k \hat{h}_t(z_i)) = \mathbb{E}(\prod_{i=1}^k (\tilde{h}_{t_i}(g(z_i)) - \arg(g'(z_i)) + \epsilon_i(t)) \mathbb{1}_{\mathcal{A}(t, z_i)}) \quad (3.23)$$

By eq. (3.8),

$$|t_i - t| \leq \log 4 + \log(1/r). \quad (3.24)$$

Therefore almost surely,  $9t/10 \leq t_i \leq 11t/10$  for all  $i$  from the choice of  $t$ . Thus  $t_i \rightarrow \infty$  as  $t \rightarrow \infty$ . Further  $|\epsilon_i(t)| = O(e^{-t/2}/r) = O(e^{-t/2+t/10}) \rightarrow 0$  for all  $i$  on the event  $\mathcal{A}(t, z_i)$  from Lemma 3.10. Using all this information, Cauchy-Schwarz, Lemmas 3.15 and 3.16, we obtain

$$|\mathbb{E}(\prod_{i=1}^k (\tilde{h}_{t_i}(g(z_i)) - \arg(g'(z_i)) + \epsilon_i(t)) \mathbb{1}_{\mathcal{A}(t, z_i)}) - \mathbb{E}(\prod_{i=1}^k (\tilde{h}_{9t/10}(g(z_i)) - \arg(g'(z_i)))| \leq ct^k e^{-c't}$$

almost surely given  $A$ . The second item of the proposition now follows from the first item.  $\square$

To prepare for the proof of convergence in the Sobolev space  $H^{-1-\eta}$  for all  $\eta > 0$  we need the following convergence of  $h_t$  integrated against test functions.

**Lemma 3.17.** *Let  $\{f_i\}_{1 \leq i \leq n}$  be smooth compactly supported functions in  $\mathbb{D}$ . Then for any sequence of integers  $k_1, \dots, k_n$ ,*

$$\lim_{t \rightarrow \infty} \mathbb{E} \prod_{i=1}^n \left( \int_{\mathbb{D}} h_t(z) f_i(z) dz \right)^{k_i} = \int_{\mathbb{D}^{\sum_{i=1}^n k_i}} H(z_{11}, \dots, z_{1k_1}, \dots, z_{n1}, \dots, z_{nk_n}) \prod_{i=1}^n \prod_{j=1}^{k_i} f_i(z_{ij}) dz_{ij}$$

where  $H$  is as in Proposition 3.13.

*Proof.* Straightforward expansion and Fubini's theorem yield

$$\mathbb{E} \prod_{i=1}^n \left( \int_{\mathbb{D}} h_t(z) f_i(z) dz \right)^{k_i} = \int_{\mathbb{D}^{\sum_{i=1}^n k_i}} \mathbb{E} \left( \prod_{i=1}^n \prod_{j=1}^{k_i} h_t(z_{ij}) \right) \prod_{i=1}^n \prod_{j=1}^{k_i} f_i(z_{ij}) dz_{ij} \quad (3.25)$$

We can apply Fubini because the term inside the integral is integrable from the moment bounds in Lemma 3.12. We want to take the limit as  $t \rightarrow \infty$  on both sides of (3.25) and apply dominated convergence theorem and Proposition 3.13 to complete the proof. To justify the application of dominated convergence theorem note that  $\mathbb{E} |\prod_{i=1}^n \prod_{j=1}^{k_i} h_t(z_{ij})| \leq \log^{\sum_{i=1}^n k_i/2}(r)$  where  $r = \min_{(i,j),(i',j')} |z_{ij} - z_{i'j'}| \wedge \min_{ij} |z_{ij} - \partial\mathbb{D}|$  via Lemma 3.16 which is integrable. Further the functions  $f_i$ 's are uniformly bounded.  $\square$

### 3.5 Identifying winding as the GFF: imaginary geometry

We need the fact that the covariance is small when one of the points is near the boundary. For this we start by a deterministic lemma about the argument of maps that remove a small set.

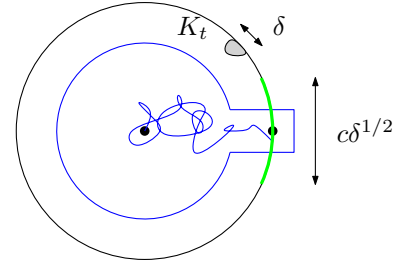
**Lemma 3.18** (distortion of argument). *Let  $K$  be a closed subset of  $\bar{\mathbb{D}}$  such that  $H = \mathbb{D} \setminus K$  is simply connected (i.e.  $K$  is a hull). Further assume that the diameter of  $K$  is smaller than some  $\delta < 1/2$  and  $1 \notin B(K, \delta^{1/2})$ . Let  $\tilde{g}$  denote the conformal map sending  $H$  to  $\mathbb{D}$  with  $\tilde{g}(0) = 0$  and  $\tilde{g}(1) = 1$ . Then  $|\arg_{\tilde{g}'(H)}(\tilde{g}'(0))| < c\delta^{1/2}$ , where  $c$  is a universal constant. Here  $\arg_{\tilde{g}'(H)}(\cdot)$  is the argument in  $\tilde{g}'(H)$  (which does not contain 0) normalised so that  $\arg_{\tilde{g}'(H)}(\tilde{g}'(1)) = 0$ .*

*Proof.* Let  $T$  be the capacity of  $K$  seen from 0 and let  $(K_t)_{t \leq T}$  be a growing sequence of hulls of capacity  $t$  such that  $K_T = K$ . Let  $g_t$  denote the Lowner maps associated to the  $K_t$ , with the usual convention  $g_t(0) = 0$  and  $g'_t(0) \in \mathbb{R}^+$  and note that  $\tilde{g} = g_T/g_T(1)$ . Let  $W : [0, \infty) \mapsto \mathbb{R}$  be the driving function for the radial Loewner differential equation (2.1) for the maps  $(g_t)_{t \geq 0}$  and  $H_t = \mathbb{D} \setminus K_t$ .

Since the conformal radius  $R(0, H_t)$  is at least the in-radius which is at least  $1 - \delta$ , we obtain that the capacity  $t$  of  $K_t$  seen from 0 is always smaller than  $-\log(1 - \delta) \leq c\delta$  for  $t \leq T$ , in particular  $T \leq c\delta$ . Let  $A_t = \partial\mathbb{D} \setminus g_t(\bar{\mathbb{D}} \setminus K_t)$ , i.e.,  $A_t$  is the part of  $\partial\mathbb{D}$  where  $g_t$  maps the boundary of  $K_t$ . By definition,  $e^{iW_t} \in A_t$  for any  $t$ . Let  $\text{Harm}_D(z, S)$  denote the harmonic measure seen from  $z$  of a set  $S$  in the domain  $D$ . Since  $\text{dist}(1, K) \geq \delta^{1/2}$ , we can find  $\theta_- < 0 < \theta_+$  such that

$$\text{Harm}_H(0, I^\pm) \geq c\delta^{1/2}, \quad I^\pm \cap K = \emptyset$$

where  $I^\pm$  are the two arcs connecting  $e^{i\theta_\pm}$  to 1 (the green arcs in Figure 7). Note that since  $K_t \subset K_T = K$ , the same holds for  $\text{Harm}_{H_t}$ . Applying conformal invariance of the harmonic measure, we get  $\text{Harm}_{\mathbb{D}}(g_t(I^\pm)) \geq c\delta^{1/2}$  and therefore  $\text{Diam}(g_t(I^\pm)) \geq c\delta^{1/2}$ . Finally since  $I^\pm$  did not intersect  $K_T$ ,  $g_t(I_\pm)$  does not intersect  $A_t$  and therefore  $|e^{iW_t} - g_t(1)| \geq c\delta^{1/2}$ .



**Figure 7:** Sketch of proof of Lemma 3.18. The probability that the Brownian motion exits through the green arc all the while staying within the region bounded by blue arcs is  $c\delta^{1/2}$ . This is a lower bound for  $\text{Harm}_{H_T}(0, I^\pm)$ .



Using this bound, the Loewner equation gives

$$|\partial_t g_t(1)| = |g_t(1) \frac{e^{iW_t} + g_t(1)}{e^{iW_t} - g_t(1)}| \leq c\delta^{-1/2}.$$

Integrating for a time  $t \leq T \leq c\delta$  gives  $|g_t(1) - 1| \leq c\delta^{1/2}$ . Recall  $\text{Arg}$  is the principal branch of argument in  $(-\pi, \pi]$ . This gives  $|\text{Arg}(g_T(1))| \leq c\delta^{1/2}$  and since by definition  $g'_t(0) \in \mathbb{R}^+$ ,

$$|\text{Arg } \tilde{g}'_t(0)| = |\text{Arg } g'_t(0) - \text{Arg } g_t(1)| \leq c\delta^{1/2}, \quad (3.26)$$

where  $\tilde{g}_t = g_t/g_t(1)$  is the conformal map sending  $H_t$  to  $\mathbb{D}$  such that  $\tilde{g}(0) = 0$  and  $\tilde{g}(1) = 1$ . Note that  $\text{Arg } \tilde{g}'_t(0)$  is continuous in  $t$  since it is bounded by  $c\delta^{1/2}$  (and so does not come into the region where  $\text{Arg}$  is discontinuous).

We deduce that the same bound as eq. (3.26) holds for  $\arg_{\tilde{g}'_T(H_T)}(0)$ . Indeed note that one can find a curve  $\lambda[0, 1]$  connecting 1 to 0 which avoids  $K_T$ . By definition,

$$\arg_{g'_t(H_t)}(0) = \int_{g'_t(\lambda)} \frac{dz}{z} = \int_0^1 \frac{g''_t(\lambda(s))\lambda'(s)}{g'_t(\lambda(s))} ds.$$

Thus  $\arg_{g'_t(H_t)}(0)$  is continuous in  $t$  since  $g_t$  has a conformal extension in a neighbourhood of  $\lambda$  which implies that all its derivatives are continuous in  $t$  in this neighbourhood. Also for  $t = 0$  the difference is 0 trivially. This concludes the proof.  $\square$

We define

$$G(z_1, z_2, \dots, z_k) = \lim_{t \rightarrow \infty} \mathbb{E} \left( \prod_{i=1}^k (h_t(z_i) - \mathbb{E}(h_t(z_i))) \right) \quad (3.27)$$

to be the  $k$ -point covariance function which exists by Proposition 3.13.

**Lemma 3.19.** *For all  $|z| \geq 3/4$  and  $t_2 \geq t_1 > -10 \log \text{dist}(z, \partial\mathbb{D}) + 1$  such that  $t_2 < 10t_1$ ,*

$$|\mathbb{E}(h_{t_1}(z)h_{t_2}(0) - \mathbb{E}(h_{t_1}(z))\mathbb{E}(h_{t_2}(0)))| \leq c \text{dist}(z, \partial\mathbb{D})^{c'}$$

In particular, as  $z \rightarrow \partial\mathbb{D}$ ,  $G(0, z) \rightarrow 0$ .

*Proof.* Let  $\delta = \text{dist}(z, \partial\mathbb{D}) = 1 - |z|$ . Set  $t = -10 \log \delta + 1$ . By Lemma 3.15,

$$|\mathbb{E}(h_{t_1}(z)h_{t_2}(0) - h_t(z)h_t(0))| \leq cte^{-c't}, \quad |\mathbb{E}h_{t_1}(z)\mathbb{E}h_{t_2}(0) - \mathbb{E}h_t(z)\mathbb{E}h_t(0)| \leq cte^{-c't}.$$

and observe that  $cte^{-c't} \leq c\delta^{c'}$ . Let us define the event  $\mathcal{G} := \mathcal{A}(t, z) \cap \{\gamma_z \subset B(z, \sqrt{\delta})\}$ . From the exponential bound on the probability of  $\mathcal{A}(t, z)$ ,  $\mathcal{A}(t, 0)$  and Lemma 3.14, we have

$$\mathbb{P}(\mathcal{G} \cap \mathcal{A}(t, 0)) \geq 1 - ce^{-c't}. \quad (3.28)$$

By Lemma 3.12 and Cauchy-Schwarz, we see that

$$|\mathbb{E}(h_t(z)h_t(0)) - \mathbb{E}(h_t(z)h_t(0)\mathbb{1}_{\mathcal{G}, \mathcal{A}(t, 0)})| \leq cte^{-c't} \quad (3.29)$$

Let  $g : \mathbb{D} \setminus \gamma_z \mapsto \mathbb{D}$  be a conformal map fixing 0 and 1. Then from Lemma 3.10, we have for some independent copy  $\tilde{h}_t(0)$  of  $h_t(0)$ ,

$$\mathbb{E}(h_t(0)\mathbb{1}_{\mathcal{A}(t, 0)} \Big| \gamma_z) \mathbb{1}_{\mathcal{G}} = \mathbb{E} \left( (\tilde{h}_s(0) - \arg_{g'(\mathbb{D} \setminus \gamma_z)}(g'(0)) + \epsilon(t)) \mathbb{1}_{\mathcal{A}(t, 0)} \right) \mathbb{1}_{\mathcal{G}} \quad (3.30)$$

where  $|\epsilon(t)| < ce^{-c't}$  on  $\mathcal{A}(t, 0)$  and  $t + \log(|z| - \sqrt{\delta}) \leq s \leq t + \log 4$  (as in eq. (3.8)). Note also that  $|z| - \sqrt{\delta} \geq 1 - 2\sqrt{\delta} > 0$ . Thus we obtain using eq. (3.29),

$$\mathbb{E}(h_t(z)h_t(0)) = O(e^{-ct}) + \mathbb{E}\left(h_t(z)\mathbb{E}\left(\tilde{h}_s(0) - \arg_{g'(\mathbb{D} \setminus \gamma_z)}(g'(0)) + \epsilon(t)\right)\mathbb{1}_{\mathcal{A}(t,0)}\middle|\gamma_z\right)\mathbb{1}_{\mathcal{G}}\right) \quad (3.31)$$

We now expand the terms in the right hand side and treat each of them separately. Observe that while  $\tilde{h}$  is independent of  $h$ ,  $s$  still depends on  $\gamma_z$ . Nevertheless, we claim that

$$\mathbb{E}(h_t(z)\mathbb{E}(\tilde{h}_s(0)|\gamma_z)) \rightarrow H(z)H(0).$$

Indeed first replace  $\tilde{h}_s(0)$  by  $\tilde{h}_{t+\log(1-2\sqrt{\delta})}(0)$  and then the above convergence is true; then using Cauchy–Schwarz and Lemma 3.15 we deduce that the above convergence holds.

Regarding the second term, we claim that  $\mathbb{E}(h_t(z) \arg_{g'(\mathbb{D} \setminus \gamma_z)}(g'(0))\mathbb{1}_{\mathcal{G}})$  converges to 0. Indeed, this follows from the distortion estimate on the argument we did in Lemma 3.18 and the fact that on  $\mathcal{G}$ ,  $\text{Diam}(\gamma_z(t)) < \sqrt{\delta}$ . Hence by Cauchy–Schwarz we conclude

$$\mathbb{E}|h_t(z) \arg_{g'(\mathbb{D} \setminus \gamma_z)}(g'(0))\mathbb{1}_{\mathcal{G}}| < cte^{-c't}.$$

Finally, for the third term, since  $|\epsilon(t)| \leq e^{-c't}$  on  $\mathcal{A}(t, 0)$ , we deduce that  $\mathbb{E}(h_t(0)\epsilon(t)\mathbb{1}_{\mathcal{A}(t,0),\mathcal{G}}) \leq te^{-ct}$  by Cauchy–Schwarz and the moment bound. Consequently, we have proved

$$|\mathbb{E}(h_t(z)h_t(0)) - H(z)H(0)| \leq ce^{-c't}.$$

Using Lemma 3.15 we deduce that  $|\mathbb{E}(h_t(z)h_t(0)) - \mathbb{E}(h_t(z))\mathbb{E}(h_t(0))| \leq ce^{-c't}$ . This proves the lemma.  $\square$

**Lemma 3.20.** *Let  $\{w_1, w_2, \dots, w_k\}$  be a set of points in  $\mathbb{D}$  all of which are distinct. Let  $A = \{\gamma_{w_1}, \dots, \gamma_{w_k}\}$  be the corresponding set of branches of  $\mathcal{T}$  in  $\mathbb{D}$ . Let  $g : \mathbb{D} \setminus A \rightarrow \mathbb{D}$  be a conformal map mapping 1 to 1. Let  $g_z : \mathbb{D} \setminus A \mapsto \mathbb{D}$  be a conformal map which maps  $z$  to 0 and 1 to 1. Then for any test function  $f$  in  $C^\infty(\mathbb{D})$ ,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E} \left[ \int_{\mathbb{D}} \hat{h}_t(z) f(z) \mathbb{1}_{\text{dist}(z,A) > e^{-t/10}} dz \middle| A \right] &= \int_{\mathbb{D}} \left( \frac{3\pi}{2} + 2 \arg_{1-\mathbb{D}}(1 - g(x)) - \arg_{g'(\mathbb{D} \setminus A)}(g'(x)) \right) f(x) dx \\ \lim_{t \rightarrow \infty} \text{Var} \left[ \int_{\mathbb{D}} \hat{h}_t(z) f(z) \mathbb{1}_{\text{dist}(z,A) > e^{-t/10}} dz \middle| A \right] &= \int_{\mathbb{D} \times \mathbb{D}} G(0, g_z(w)) f(z) f(w) dz dw \end{aligned} \quad (3.32)$$

almost surely, where  $G(\cdot, \cdot)$  is the two-point covariance function defined in eq. (3.27).

*Proof.* This proof is an application of dominated convergence theorem. For the first item, note that for a fixed  $t$  we can take the expectation inside by Fubini and the moment bounds of  $\hat{h}_t$ . Again observe that from (3.23), we have for an independent copy  $\tilde{h}_t$  of  $h_t$  in  $\mathbb{D}$ ,

$$\begin{aligned} \int_{\mathbb{D}} \mathbb{E} \left[ \hat{h}_t(z) f(z) \mathbb{1}_{\text{dist}(z,A) > e^{-t/10}} dz \middle| A \right] \\ = \int_{\mathbb{D}} \mathbb{1}_{\text{dist}(z,A) > e^{-t/10}, \mathcal{A}(t,z)} \mathbb{E} \left( \tilde{h}_{t'}(z) - \arg_{g'(\mathbb{D} \setminus A)} g'(z) \middle| A \right) f(z) dz + O(e^{-ct}) \end{aligned} \quad (3.33)$$

where  $9t/10 < t' = t'(z) < 11t/10$  almost surely by Lemma 3.10. Therefore, the first item follows by taking limit on both sides and using dominated convergence theorem (whose application is justified by say Lemma 3.16).

For the variance computation, write  $\mathbb{E}^A$  for conditional expectation given  $A$ . Then observe that

$$\begin{aligned} & \mathbb{E}^A \left[ \left( \hat{h}_t(z) - \mathbb{E}^A \hat{h}_t(z) \right) \mathbb{1}_{\text{dist}(z,A) > e^{-t/10}} \left( \hat{h}_t(w) - \mathbb{E}^A \hat{h}_t(w) \right) \mathbb{1}_{\text{dist}(w,A) > e^{-t/10}} \right] \\ &= \mathbb{E}^A \left[ \prod_{x \in \{z,w\}} \left( \tilde{h}_{t_x}(g_z(x)) + \epsilon_x(t) - \mathbb{E} \tilde{h}_{t_x}(g_z(x)) \right) \mathbb{1}_{\mathcal{A}(t,x); \text{dist}(x,A) > e^{-t/10}} \right] \end{aligned} \quad (3.34)$$

because once we condition on  $A$ , the term  $\arg_{g'_z(\mathbb{D} \setminus A)}(g'_z(x))$  is nonrandom and hence cancels out. Note that since  $z, w$  are at least at a distance  $e^{-t/10}$  away from  $A$ , we have  $9t/10 \leq t_x \leq 11t/10$  for  $x \in \{z, w\}$ , and that  $|\epsilon_x(t)| \leq ce^{-c't}$  on  $\mathcal{A}(t, x)$ . By Cauchy–Schwarz and Lemmas 3.12 and 3.15, note that in the right hand side we can replace  $t_z, t_w$  by  $t$  provided that we add an error term bounded by  $ce^{-c't}$ , uniformly in  $z$  and  $w$ .

Hence, by Fubini and eq. (3.34),

$$\begin{aligned} & \text{Var} \left[ \int_{\mathbb{D}} \hat{h}_t(z) f(z) \mathbb{1}_{\text{dist}(z,A) > e^{-t/10}} dz \middle| A \right] \\ &= \int_{\mathbb{D}^2} \mathbb{E} \left[ \prod_{x \in \{z,w\}} \left( \tilde{h}_t(g_z(x)) + \epsilon_x(t) - \mathbb{E} \tilde{h}_t(g_z(x)) \right) \mathbb{1}_{\mathcal{A}(t,x); \text{dist}(x,A) > e^{-t/10}} \right] f(z) f(w) dz dw + \text{error}(t) \end{aligned} \quad (3.35)$$

$$\begin{aligned} &= \int_{\mathbb{D}^2} \text{Cov} \left( \tilde{h}_t(0), \tilde{h}_t(g_z(w)) \right) f(z) f(w) dz dw + \text{error}(t) \\ &= \int_{z \in \mathbb{D}} f(z) \int_{y \in \mathbb{D}} \text{Cov} \left( \tilde{h}_t(0), \tilde{h}_t(y) \right) |(g_z^{-1})'(y)|^2 f(g_z^{-1}(y)) dy dz + \text{error}(t) \end{aligned} \quad (3.36)$$

where the error term satisfies  $|\text{error}(t)| \leq cte^{-c't}$ .

By Lemma 3.19, one can find a  $\delta$  such that for all  $y \in \mathbb{D}$  with  $|y| > 1 - \delta$  and all  $t > -12 \log \delta + 1$ ,  $|\text{Cov}(\tilde{h}_t(0), \tilde{h}_t(y))| < 1$  almost surely. Therefore, on the set  $\{|y| \geq 1 - \delta\}$ , the integrand in eq. (3.36) is bounded by  $a(y, z) := |(g_z^{-1})'(y)|^2 \|f\|_{\infty}^2$ . On the other hand, by Lemma 3.16 the integrand is bounded by  $b(y, z) := \log(|y| \wedge (1 - |y|)) |(g_z^{-1})'(y)|^2 \|f\|_{\infty}^2$  when  $|y| \leq 1 - \delta$ .

Note that

$$\int_{\{|y| > 1 - \delta\}} |(g_z^{-1})'(y)|^2 dy = \text{Leb}(y : g_z^{-1}(\{|y| > 1 - \delta\})) \leq \pi.$$

Therefore  $a(y, z)$  is integrable on  $\{|y| > 1 - \delta\}$ . Note also that if  $|y| \leq 1 - \delta$ ,  $|(g_z^{-1})'(y)| = R(g_z^{-1}(y), g_z^{-1}(\mathbb{D}))/R(y, \mathbb{D}) < c\delta^{-1}$ , so  $b$  too is integrable on  $\{|y| \leq 1 - \delta\}$ . Thus we can take limit inside the integral. Finally, one can use Proposition 3.13, item 1 to conclude.  $\square$

We are now going to use the imaginary geometry coupling of Theorem 2.6 to prove the following consequence.

**Theorem 3.21.** *Let  $f$  be any test function in  $C^\infty(\bar{\mathbb{D}})$ . Let  $h = h_{\text{GFF}}^0 + \chi u_{\mathbb{D},1}$  be the Gaussian free field coupled with the UST according to Theorem 2.6, and let  $h_{\text{GFF}} = 1/\chi h + \pi/2$ . Then we have*

$$\lim_{t \rightarrow \infty} \mathbb{E} \left( (h_t, f) - (h_{\text{GFF}}, f) \right)^2 = 0$$

In particular,  $(h_t, f)$  converges to  $(h_{\text{GFF}}, f)$  in  $L^2(\mathbb{P})$  and in probability as  $t \rightarrow \infty$ .

*Proof.* Fix  $\varepsilon > 0$ . Using Lemma 3.19, pick  $\delta$  such that  $G(0, y) + 2 \log(1/|y|) < \varepsilon$  if  $|y| \in (1 - \delta, 1)$ . Fix  $\eta$  to be chosen suitably later (in a way which is allowed to depend on  $\varepsilon$  and  $\delta$ ). Let  $A$  be the set of branches of  $\mathcal{T}$  from a “dense” set of points,  $\frac{\eta}{4}\mathbb{Z}^2 \cap \mathbb{D}$ , to 1. Note that that  $R(z, \mathbb{D} \setminus A) < \eta$  for any  $z \in \mathbb{D}$  by Koebe’s 1/4 theorem. Let  $D' = \mathbb{D} \setminus A$ . Define  $\bar{h}_t(z) = \hat{h}_t \mathbb{1}_{\text{dist}(z, A) > e^{-t/10}}$ . First of all notice that

$$\mathbb{E}((h_t, f) - (h_{\text{GFF}}, f))^2 \leq 2\mathbb{E}((\bar{h}_t, f) - (h_{\text{GFF}}, f))^2 + 2\mathbb{E}((\bar{h}_t, f) - (h_t, f))^2 \quad (3.37)$$

By adding and removing  $\hat{h}_t$  the last expression on the right hand side of (3.37) converges to 0 by (3.22) and the following fact: using Cauchy–Schwarz and the moment bounds on  $\hat{h}_t(z)$ ,

$$\begin{aligned} \mathbb{E}((\bar{h}_t, f) - (\hat{h}_t, f))^2 &= \mathbb{E}\left(\int_{\mathbb{D}} \hat{h}_t(z) f(z) \mathbb{1}_{\text{dist}(z, A) \leq e^{-t/10}} dz\right)^2 \\ &\leq \int_{\mathbb{D}^2} \mathbb{E}|\hat{h}_t(z) \hat{h}_t(w)| \|f\|_{\infty}^2 \mathbb{1}_{\text{dist}(z, A) \vee \text{dist}(w, A) \leq e^{-t/10}} dz dw \\ &\leq c \|f\|_{\infty}^2 (1+t) \int_{\mathbb{D}^2} \mathbb{P}(\text{dist}(z, A) \vee \text{dist}(w, A) \leq e^{-t/10})^{1/2} dz dw \\ &\leq c \|f\|_{\infty}^2 (1+t) \mathbb{P}(\text{dist}(U_1, A) \vee \text{dist}(U_2, A) \leq e^{-t/10})^{1/2} \end{aligned}$$

where  $U_i \sim \text{Unif}(\mathbb{D})$  and are independent of everything else and each other. Now if  $z$  is a point in  $\frac{\eta}{4}\mathbb{Z}^2 \cap \mathbb{D}$ , then

$$\mathbb{P}(\text{dist}(U, \gamma_z) \leq e^{-t/10}) \leq c_{\eta} e^{-c't}$$

by Lemma 3.14. Hence summing up over  $O(1/\eta^2)$  points and using a union bound, we see that (for every fixed  $\eta$ ),  $\mathbb{P}(\text{dist}(U_1, A) \vee \text{dist}(U_2, A) \leq e^{-t/10})^{1/2} \rightarrow 0$  exponentially fast and thus the second term on the right hand side of eq. (3.37) tends to 0.

Let  $\mathbb{E}^A$  and  $\text{Var}^A$  denote the conditional expectation and variance given  $A$ . It is easy to see

$$\mathbb{E}^A((\bar{h}_t, f) - (h_{\text{GFF}}, f))^2 \leq 3\text{Var}^A(\bar{h}_t, f) + 3\text{Var}^A(h_{\text{GFF}}, f) + 3(\mathbb{E}^A(\bar{h}_t, f) - \mathbb{E}^A(h_{\text{GFF}}, f))^2 \quad (3.38)$$

Note that it is enough to show that as  $t \rightarrow \infty$  the left hand side of (3.38) can be made smaller than  $\varepsilon$  (in expectation) by choosing  $\eta$  suitably since this implies that the first term in the right hand side of (3.37) is smaller than  $\varepsilon$  plus a term converging to zero, which completes the proof.

The last term of (3.38) converges to 0 from the convergence of expectations in Lemma 3.20 and the fact that  $h_{\text{GFF}}$  satisfies the correct boundary conditions given  $A$  (which is a consequence of the imaginary geometry coupling).

For the other terms, recall that conditionally on  $A$ ,  $h_{\text{GFF}}$  is just a free field in  $D'$  with variance  $1/\chi^2$  and with Dirichlet boundary condition plus a harmonic function. Recall that the variance of a GFF integrated against a test function is given by an integral of the Green’s function in the domain. Also recall that the Green’s function is conformally invariant. In particular if  $g_z$  is the conformal map from  $D'$  to  $\mathbb{D}$  sending  $z$  to 0 and 1 to 1, using the explicit formula for the Green’s function in the unit disc, we have

$$\text{Var}^A(h_{\text{GFF}}) = - \int_{\mathbb{D} \times \mathbb{D}} \frac{1}{\chi^2} \log |g_z(w)| f(z) f(w) dz dw.$$

Plugging in the variance formula derived in Lemma 3.20 and since  $\chi = 1/\sqrt{2}$ ,

$$\mathbb{E}(\text{Var}^A((h_{\text{GFF}}, f))) + \mathbb{E}(\lim_{t \rightarrow \infty} \text{Var}^A(\bar{h}_t, f)) = \mathbb{E}\left(\int_{\mathbb{D} \times \mathbb{D}} (G(0, g_z(w)) - 2 \log |g_z(w)|) f(z) f(w) dz dw\right) \quad (3.39)$$

By a change of variable  $y = g_z(w)$ ,

$$\int_{\mathbb{D}} (G(0, g_z(w)) - 2 \log |g_z(w)|) f(w) dw = \int_{\mathbb{D}} (G(0, y) - 2 \log |y|) f(g_z^{-1}(y)) |(g_z^{-1})'(y)|^2 dy \quad (3.40)$$

As in Lemma 3.20, we are going to estimate the integral on two domains,  $B_\delta = \{|z| < 1 - \delta\}$  and  $B'_\delta := \mathbb{D} \setminus B_\delta$ . Recall that by the choice of  $\delta$ ,  $G(0, y) + 2 \log 1/|y| < \varepsilon$  if  $y \in B'_\delta$ . Hence

$$\int_{B'_\delta} |(G(0, y) - 2 \log |y|) f(g_z^{-1}(y))| |(g_z^{-1})'(y)|^2 dy < \varepsilon \|f\|_\infty \text{Leb}(g_z^{-1}(B'_\delta)) < \pi \varepsilon \|f\|_\infty$$

To estimate the integral in  $B_\delta$ , notice that  $|(g_z^{-1})'(y)| = R(g_z^{-1}(y), D')/R(y, D) < \eta/\delta$  if  $y \in B_\delta$ , by Koebe's 1/4 theorem. Hence

$$\int_{B_\delta} |(G(0, y) - 2 \log |y|) f(g_z^{-1}(w))| |(g_z^{-1})'(y)|^2 dw < \|f\|_\infty \frac{\eta^2}{\delta^2} \int_{B_\delta} |G(0, y) - 2 \log |y|| dy \quad (3.41)$$

The integral on the right hand side is finite via the bound Lemma 3.16. After bounding the integral over  $w$ , we bound the integral over  $z$  by  $\|f\|_\infty$  times the area. This completes the proof since we can choose  $\eta$  such that  $\eta/\delta < \varepsilon$  where  $\varepsilon$  is arbitrary.  $\square$

**Corollary 3.22.** *For any  $p > 0$ , and any sequence of test functions  $(f_1, \dots, f_n) \in C^\infty(\bar{\mathbb{D}})$  and integers  $k_1, \dots, k_n$ ,*

$$\mathbb{E} \left( \prod_{i=1}^n (h_t, f_i)^{k_i} - \prod_{i=1}^n (h_{\text{GFF}}, f_i)^{k_i} \right)^p \rightarrow 0 \quad (3.42)$$

as  $t \rightarrow \infty$ .

*Proof.* It is enough to prove this fact when  $p$  is an integer. For  $n = 1$  this follows from the fact that  $(h_t, f)$  converges in  $L^2(\mathbb{P})$  towards  $(h_{\text{GFF}}, f)$ , and  $(h_t, f)$  is bounded in  $L^p$  for any  $p > 1$ .

For general  $n \geq 1$  we proceed by induction, and note that by the triangle inequality in  $L^p$  (i.e., Minkowski's inequality), if  $X_n \rightarrow X$  and  $Y_n \rightarrow Y$  in  $L^p$  for every  $p > 1$  then  $X_n Y_n \rightarrow XY$  in  $L^p$  for every  $p > 1$ .  $\square$

### 3.6 General domains

In this section we prove our result when  $D$  is a bounded domain with a locally connected boundary. Recall that our definition of  $u_{(D,x)}$  in (2.9) only makes sense when the boundary is smooth in a neighbourhood of a marked point  $x \in \partial D$  (while otherwise it is only defined up to a global additive constant, see Remark 2.5). The general idea is to show that in the limit one has

$$h^D \circ \varphi(z) = h^{\mathbb{D}}(z) + \arg_{\varphi'(\mathbb{D})}(\varphi'(z)), \quad z \in \mathbb{D}$$

which is the imaginary geometry change of coordinates (see [34, 35]).

**Theorem 3.23.** *Let  $D$  be as above. Let  $f$  be any bounded Borel test function defined on  $\bar{D}$ . Let  $h = h_{\text{GFF}}^0 + \chi u_{(D,x)}$  be the GFF coupled to the UST according to the imaginary geometry coupling of Theorem 2.6 and  $u_{(D,x)}$  is as in (2.9). Then  $(h_t^D, f)$  converges to  $(h_{\text{GFF}}^D, f)$  in  $L^2(\mathbb{P})$  and in probability as  $t \rightarrow \infty$ , where  $h_{\text{GFF}}^D = \chi^{-1}h + \pi/2$ .*

*Proof.* Let  $h_t^D$  be the winding field in  $D$  as defined as in (3.1) or (3.2) as appropriate. Let  $\mathcal{T}^D$  denote the continuum wired UST in  $D$ . Fix a conformal map  $\varphi : \mathbb{D} \rightarrow D$  sending the marked point 1 to the marked point  $x \in \partial D$ . Let  $h^\mathbb{D}$  denote the intrinsic winding field associated to the continuum spanning tree  $\mathcal{T} = \varphi^{-1}(\mathcal{T}^D)$ .

Let  $\tilde{\gamma}_z(t)$  denote the branch of the UST towards  $z$  in  $\mathcal{T}^D$ . Let  $\mathcal{A}^D(t, z)$  be the event that  $|\tilde{\gamma}_z(t) - z| < e^{-t/2}R(z, D)$ . Applying Koebe's 1/4 theorem, we conclude using Theorem 3.1 that  $\mathbb{P}(\mathcal{A}^D(t, z)) \geq 1 - e^{-ct}$ .

First note that in the smooth case, using Corollary 3.7 (it is clear that we can apply this since we are dealing with conformal images of continuous curves in  $\mathbb{D}$  and we can also extend  $\arg \varphi'$  to 1 using Lemma 3.9), we conclude that

$$h_t^D \circ \varphi(z) 1_{\mathcal{A}^D(t, z)} = \left( h_t^\mathbb{D}(z) + \arg_{\varphi'(\mathbb{D})}(\varphi'(z)) + \epsilon \right) 1_{\mathcal{A}^D(t, z)} \quad (3.43)$$

where  $|\epsilon| < e^{-ct}$ . In the general case, using Corollary 3.7, we get the same equation (3.43) up to a global constant. Using this and eq. (3.14), we immediately conclude that

$$\mathbb{E}((h_t^D \circ \varphi(z) - \arg_{\varphi'(\mathbb{D})}(\varphi'(z)))^2 1_{\mathcal{A}^D(t, z)}) \leq C(1 + t).$$

To bound the moment on the complement of  $\mathcal{A}(t, z)$ , unfortunately we need to use the estimate in Proposition 4.10 on discrete loop erased walk that we prove later. This proposition is telling us that the maximal topological winding in an interval of length 1 has exponential tail. To go from discrete to the continuum, we can use Proposition 4.10 on the square lattice, let the mesh size go to 0 and use Lawler, Schramm and Werner [31].

Note that the curve  $\tilde{\gamma}_z[-1, t]$  must come within distance  $e^{-t/2}R(z, D)$  of  $z$ . So by Koebe's 1/4 theorem there is a (random) time  $t'$  satisfying  $t/2 - \log 4 < t' < t/2$  so that  $|\tilde{\gamma}_z(t') - z| < e^{-t/2}R(z, D)$ . Now observe that

$$h_t^D \circ \varphi(z) = h_{t'}^D \circ \varphi(z) + W(\tilde{\gamma}_z[t', t]) \quad (3.44)$$

Using eq. (3.43) and applying Proposition 4.10 to  $h_{t'}^\mathbb{D}$  we see that the second moment of  $h_{t'}^D \circ \varphi(z) - \arg_{\varphi'(\mathbb{D})}(\varphi'(z))$  is bounded by  $C(1 + t)$ . Using the additivity of the winding, the second moment of the winding of  $\tilde{\gamma}_z[t', t]$  is bounded by  $C(1 + t - t')$  again using Proposition 4.10, but this time in  $D$ . Thus overall, we get

$$\mathbb{E}\left((h_t^D \circ \varphi(z) - \arg_{\varphi'(\mathbb{D})}(\varphi'(z)))^2\right) \leq C(1 + t) \quad (3.45)$$

Also in the general case, (3.45) is valid up to a global constant (meaning that for every choice of constant in the left hand side, the inequality holds for an appropriate choice of  $C$  in the right hand side). Now note that

$$\mathbb{E}(h_t^D \circ \varphi - h_{\text{GFF}}^D \circ \varphi, f \circ \varphi)^2 \leq 2\mathbb{E}(h_t^D \circ \varphi - h_t^\mathbb{D} - \arg_{\varphi'(\mathbb{D})} \varphi', f \circ \varphi)^2 + 2\mathbb{E}(h_t^\mathbb{D} - h_{\text{GFF}}^\mathbb{D}, f \circ \varphi)^2 \quad (3.46)$$

The second term on the right hand side of (3.46) converges to 0 via Theorem 3.21. The first term can be written as

$$\begin{aligned} & \mathbb{E}(h_t^D \circ \varphi - h_t^\mathbb{D} - \arg_{\varphi'(\mathbb{D})} \varphi', f \circ \varphi)^2 \\ &= \mathbb{E} \int_{\mathbb{D}^2} (h_t^D \circ \varphi(z) - h_t^\mathbb{D}(z) - \arg_{\varphi'(\mathbb{D})} \varphi'(z)) (h_t^D \circ \varphi(w) - h_t^\mathbb{D}(w) - \arg_{\varphi'(\mathbb{D})} \varphi'(w)) f \circ \varphi(z) f \circ \varphi(w) dz dw \end{aligned} \quad (3.47)$$

Each term in the integrand converges to 0 on  $\mathcal{A}^D(t, z)$  using (3.43). In the general case, the integrand converges to 0 for the choice of constant corresponding to (3.43). On the complement of  $\mathcal{A}^D(t, z)$ , the integrand also converges to 0. To see this we use the fact that the probability of  $\mathcal{A}^D(t, z)$  is exponentially high, the moment bound on (3.45) and Cauchy–Schwarz. Finally, let  $r(z, w) = R(z, D) \wedge R(w, D) \wedge |z - w|$ . Using moment bound (3.45) and Cauchy–Schwarz, we see that for  $t > -10 \log r(z, w)$ , the integrand is bounded by  $C(1 + t)e^{-ct}$  and for  $t < -10 \log r(z, w)$ , the integrand is bounded by  $C(1 + \log r(z, w))$ . Since  $\log r(z, w)$  is integrable, we conclude using dominated convergence theorem.  $\square$

**Remark 3.24.** We point out that the integral of  $\arg_{\varphi'(\mathbb{D})}(\varphi'(\cdot))$  might be infinite even in the smooth case. Also for the joint convergence of the moments, we do not need the domain to be bounded.

### 3.7 Convergence in $H^{-1-\eta}$

Let  $D$  be a domain with locally connected boundary and now assume also that  $D$  is bounded. Let  $(e_j)_{j \geq 1}$  denote the orthonormal basis of  $L^2(D)$  given by the eigenfunctions of  $-\Delta$  in  $D$ . Let  $h_{\text{GFF}}$  denote the process defined in Theorem 3.21, which is  $(1/\chi)$  times a GFF with winding boundary conditions multiplied by  $\chi$ . We now strengthen the convergence from a convergence in probability or  $L^2(\mathbb{P})$  for finite-dimensional marginals to a convergence in the Sobolev space  $H^{-1-\eta}$ .

**Proposition 3.25.** *For every  $\eta > 0$ , the field  $h_t$  converges to  $h_{\text{GFF}}$  in  $H^{-1-\eta}$  in probability as  $t \rightarrow \infty$ . Further,  $\{h_n\}_{n \in \mathbb{N}}$  converges almost surely to  $h_{\text{GFF}}$  as  $n \rightarrow \infty$  along positive integers. Also for all  $1 \leq k < \infty$ ,  $\mathbb{E}[\|h_u - h_\infty\|_{H^{-1-\eta}}^k] \rightarrow 0$ .*

*Proof.* The basic idea is to show that  $h_t$  is a Cauchy sequence in  $H^{-1-\eta}$ . Let  $u \geq t$ . We start by getting bounds on  $\mathbb{E}[(h_u - h_t, e_j)^2]$ . By Fubini's theorem and Cauchy–Schwarz,

$$(\mathbb{E}[(h_u - h_t, e_j)^2])^2 = \left( \int_{D^2} \mathbb{E}[(h_u(z) - h_t(z))(h_u(w) - h_t(w))] e_j(z) e_j(w) dz dw \right)^2 \quad (3.48)$$

$$\begin{aligned} &\leq \int_{D^2} \left( \mathbb{E}[(h_u(z) - h_t(z))(h_u(w) - h_t(w))] \right)^2 dz dw \int_{D^2} e_j^2(z) e_j^2(w) dz dw \\ &= \int_{D^2} \left( \mathbb{E}[(h_u(z) - h_t(z))(h_u(w) - h_t(w))] \right)^2 dz dw \end{aligned} \quad (3.49)$$

since  $e_j$  forms an orthonormal basis of  $L^2(D)$ . Let  $r(z, w) = |z - w| \wedge \text{dist}(z, \partial \mathbb{D}) \wedge \text{dist}(w, \partial \mathbb{D})$ . We are going to break up the integral in (3.49) into two cases, either  $t \leq -10 \log(r(z, w)) + 1$  (i.e.,  $r^{10} \leq e^{1-t}$ ) or otherwise. In the first case Cauchy–Schwarz and the bound on moment of order two using (3.43) and (3.45) yield

$$\begin{aligned} &\int_{D^2} \left( \mathbb{E}[(h_u(z) - h_t(z))(h_u(w) - h_t(w))] \right)^2 \mathbb{1}_{t \leq -10 \log(r(z, w)) + 1} dz dw \\ &\leq c(1 + u^2) \int_{D^2} \mathbb{1}_{r(z, w)^{10} \leq e^{1-t}} dz dw \leq c(1 + u^2) e^{-c't} \end{aligned} \quad (3.50)$$

On the other hand,

$$\begin{aligned} &\int_{D^2} \left( \mathbb{E}[(h_u(z) - h_t(z))(h_u(w) - h_t(w))] \right)^2 \mathbb{1}_{t > -10 \log(r(z, w)) + 1} dz dw \\ &\leq cu^2 e^{-ct} \int_{D^2} \mathbb{1}_{t > -10 \log(r(z, w)) + 1} dz dw \leq c(1 + u^2) e^{-c't} \end{aligned} \quad (3.51)$$



where the second inequality above follows from Lemma 3.15 and eq. (3.43) (note that the bound of Lemma 3.15 holds also if  $\hat{h}$  is replaced by  $h$ , because of the control on moments of  $h_t$  in Lemma 3.12 and the exponential bound on the probability of  $\mathcal{A}(t, z)^c$ ). Combining (3.50) and (3.51), we obtain

$$\mathbb{E}[(h_u - h_t, e_j)^2] \leq c(1 + u^2)e^{-ct} \quad (3.52)$$

Now let  $t = n$  and  $n \leq u \leq n + 1$ . Then by Jensen's inequality

$$[\mathbb{E}\|h_u - h_n\|_{H^{-1-\eta}}]^2 \leq \mathbb{E}[\|h_u - h_n\|_{H^{-1-\eta}}^2] = \sum_{j \geq 1} \mathbb{E}[(h_u - h_n, e_j)^2] \lambda_j^{-1-\eta} \leq c(1 + n^2)e^{-c'n} \sum_{j=1}^{\infty} \lambda_j^{-1-\eta} \quad (3.53)$$

and since  $\sum_{j=1}^{\infty} \lambda_j^{-1-\eta} < \infty$  by Weyl's law we deduce (by applying Markov's inequality and the Borel–Cantelli lemma) that  $h_n$  is almost surely a Cauchy sequence in  $H^{-1-\eta}$  along the integers, and hence converge to a limit  $h_{\infty}$  almost surely in  $H^{-1-\eta}$ . Furthermore by the triangle inequality and (3.53) we get  $\mathbb{E}(\|h_{\infty} - h_n\|_{H^{-1-\eta}}) \leq Ce^{-c'n}$ . Then using (3.53) again, we deduce

$$\mathbb{E}[\|h_u - h_{\infty}\|_{H^{-1-\eta}}] \leq C(1 + u)e^{-c'u}$$

and hence  $h_u$  converges in probability in  $H^{-1-\eta}$  to  $h_{\infty}$ . To get convergence of  $\mathbb{E}[\|h_u - h_{\infty}\|_{H^{-1-\eta}}^k]$  for any  $k \geq 1$  a similar argument would work: one needs to consider  $\mathbb{E}(h_t - h_u, e_j)^{2k}$  and hence there would be  $k$  terms inside the integral (3.48). We skip this here because an exact similar argument with minor modifications is done in the proof of Theorem 5.1 later. Furthermore we have that  $h_{\infty} = h_{\text{GFF}}$  by considering the action on test functions and Theorem 3.21. This finishes the proof of Proposition 3.25 and hence also of Theorem 3.2.  $\square$

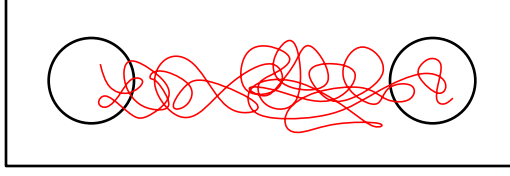
## 4 Discrete estimates on uniform spanning trees

The goal of this section is to gather the lemmas needed in Section 5 for the proof of the main result, Theorem 1.2. To make the purpose of the results in this section more clear, it will be useful for the reader to recall the general overview of the proof in Section 1.5. Recall that one additional difficulty comes from the fact that we need to work with moments and not only in law. Therefore we also need a priori estimates on the tails of our variables to use Cauchy–Schwarz bounds and dominated convergence theorems. In particular a bound on the tail of the winding of loop-erased random walk is derived in Section 4.3.

### 4.1 Assumptions on the graph.

Let  $G$  be a planar infinite graph embedded properly in the plane. This means that the embedding is such that no two edges cross each other. Vertices of the graph are identified with some points in  $\mathbb{C}$  given by the embedding. We allow  $G$  to have oriented edges with weights and a continuous simple random walk  $\{X_t\}_{t \geq 0}$  on such a graph  $G$  is defined in the usual way: the walker jumps from  $u$  to  $v$  at rate  $w(u, v)$  where  $w(u, v)$  denotes the weight of the oriented edge  $(u, v)$ . Given a vertex  $u$  in  $F$ , let  $\mathbb{P}_u$  denote the law of continuous time simple random walk on  $G$  started from  $u$ . Let  $G^{\# \delta}$  denote the embedding rescaled by  $\delta$  (this means we look at the embedding which is  $\delta$  times the original embedding). For  $A \subset \mathbb{C}$ , we denote by  $A^{\# \delta}$  the set of vertices of  $G^{\# \delta}$  in  $A$ .

In this section,  $B(a, r)$  will denote the set  $\{z : |z - a| < r\}$ . For  $A \subset \mathbb{C}$ , denote by  $A + z := \{z + x : x \in A\}$  to be the translation of  $A$  by  $z$ . We assume  $G$  has the following properties.



**Figure 8:** An illustration of the crossing condition.

- (i) **(Bounded density)** There exists  $C$  such that for any  $x \in \mathbb{C}$ , the number of vertices of  $G$  in the square  $x + [0, 1]^2$  is smaller than  $C$ .
- (ii) **(Good embedding)** The edges of the graph are embedded in such a way that they do not cross each other and have uniformly bounded winding. Also,  $0$  is a vertex.
- (iii) **(Irreducible)** The continuous time random walk on  $G$  is irreducible in the sense that for any two vertices  $u$  and  $v$  in  $G$ ,  $\mathbb{P}_u(X_1 = v) > 0$ .
- (iv) **(Invariance principle)** As  $\delta \rightarrow 0$ , the continuous time random walk  $\{X_t\}_{t \geq 0}$  on  $G^{\#\delta}$  started from  $0$  satisfies:

$$(\delta X_{t/\delta^2})_{t \geq 0} \xrightarrow[\delta \rightarrow 0]{(d)} (B_{\phi(t)})_{t \geq 0}$$

where  $(B_t, t \geq 0)$  is a two dimensional standard Brownian motion in  $\mathbb{C}$  started from  $0$ , and  $\phi$  is a nondecreasing, continuous, possibly random function satisfying  $\phi(0) = 0$  and  $\phi(\infty) = \infty$ . The above convergence holds in law in Skorokhod topology.

- (v) **(Uniform crossing estimate).** Let  $R$  be the horizontal rectangle  $[0, 3] \times [0, 1]$  and  $R'$  be the vertical rectangle  $[0, 1] \times [0, 3]$ . Let  $B_1 := B((1/2, 1/2), 1/4)$  be the *starting ball* and  $B_2 := B((5/2, 1/2), 1/4)$  be the *target ball* (see Figure 8). There exists universal constants  $\delta_0 > 0$  and  $\alpha_0 > 0$  such that for all  $z \in \mathbb{C}$ ,  $\delta \in (0, \delta_0)$ ,  $v \in B_1$  such that  $v + z \in G^{\#\delta}$ ,

$$\mathbb{P}_{v+z}(X \text{ hits } B_2 + z \text{ before exiting } (R + z)^{\#\delta}) > \alpha_0. \quad (4.1)$$

The same statement as above holds for crossing from right to left, i.e., for any  $v \in B_2$ , (4.1) holds if we replace  $B_2$  by  $B_1$ . Also, the same statement holds for the vertical rectangle  $R'$ . Let  $B'_2 = B((1/2, 5/2), 1/4)$ . Then for all  $z \in \mathbb{C}$ ,  $\delta \in (0, \delta_0)$ ,  $v \in B_1$  such that  $v + z \in G^{\#\delta}$ ,

$$\mathbb{P}_{v+z}(X \text{ hits } B'_2 + z \text{ before exiting } (R' + z)^{\#\delta}) > \alpha_0.$$

Again, the same statement holds for crossing from top to bottom, i.e., from  $B'_2$  to  $B_1$ .

Let us remark here that the crossing estimate is not a consequence of the invariance principle since we do not have a rate of convergence which is uniform over all points in the plane. Also note that the crossing estimates hold for rectangles  $R, R'$  rescaled by  $r > 0$  and translated by  $z$  for the same  $\alpha_0$  and  $\delta \in (0, r\delta_0)$ . We emphasise here that it is enough to assume the invariance principle starting from a single point.

**Remark 4.1.** A consequence of the results of Yadin and Yehudayoff [43] is that under these assumptions loop-erased random walk in  $D^{\#\delta}$ , started from an arbitrary vertex  $z^\delta \rightarrow z$  as  $\delta \rightarrow 0$ , will converge (for the uniform topology on compacts) towards radial SLE<sub>2</sub> in  $D$  towards  $z$ . Note that this holds even if  $z$  is distinct from  $0$ . There are a few details worth mentioning here:

1. Yadin and Yehudayoff [43] considered *backward* loop erased random walk, whereas we consider forward LERW (where loops are erased in chronological order). However, these have the same law, even when the graph is directed as is the case here (see [30]).
2. The result of [43] is stated with  $D = \mathbb{D}$  and  $z = 0$ , but this does not play a role in the proof. Notice in particular that the key estimate on the Poisson kernel ([43], Lemma 1.2) is stated with the generality we require. See also [43], Proposition 6.4 for a statement about the convergence of the driving function to Brownian motion in the general setup we require. Note that planarity of the graph plays a crucial role to prove this estimate. Also, the proof of tightness in the sense of Lemma 6.17 in [43] also follows through in our situation with no significant modification.

**Remark 4.2.** Let us briefly discuss the role of these assumptions. The invariance principle should be essentially a minimal assumption for the convergence. Indeed the Gaussian free field depends on the Euclidean structure of the plane and it is difficult to imagine any graph converging in a sense to the Euclidean plane without satisfying an invariance principle. In practice the invariance principle and irreducibility, together with the fact that there is no accumulation point, are exactly the assumptions needed for the convergence of the loop-erased random walk to  $\text{SLE}_2$  from [43].

Our main additional assumption is the uniform crossing estimate. It is used extensively to derive various a-priori estimates on the behaviour of the random walk, the uniformity over starting points and scale being a key factor for different multi-scale arguments. We believe however that there should be some room in our proofs to weaken this assumption, for example only assuming that “good” rectangles form a supercritical percolation.

The bounded density assumption is actually only needed for a union bound in the proof of Lemma 4.16. It is clear from that proof that it would not be needed if the uniform crossing assumption was allowed to “scale” with the local density of the graph.

## 4.2 Russo–Seymour–Welsh type estimates

Let  $D$  be a domain with locally connected boundary. To define the wired UST in the discrete domain, we perform the following surgery. For every oriented edge  $(xy)$  which intersect  $\partial D$ , we add an extra auxiliary vertex at the first intersection point (when following  $(xy)$ ). We then replace  $(xy)$  by an oriented edge from  $x$  to this auxiliary vertex, keeping the same weight. The *wired graph* is the graph induced by all the vertices in  $G^{\#\delta}$  along with all the auxiliary vertices and then wiring (or gluing) together all the auxiliary vertices. We denote by  $\partial D^{\#\delta}$  all the edges with one endpoint being an auxiliary vertex and another endpoint inside  $D$ . The wired UST  $\mathcal{T}^{\#\delta}$  is defined to be a uniform spanning tree on the wired graph. It is useful to think of the wired tree being sampled by Wilson’s algorithm with the wired vertex being the initial root vertex. All the results in this section hold without the assumption on CLT (just assumptions **i** and **v** from Section 4.1 are needed).

We denote by  $A(x, r, R)$  the annulus  $\{z \in \mathbb{C} : r < |z - x| < R\}$ . Let  $v \in A(x, r, R)^{\#\delta}$ . The random walk trajectory from a vertex  $v$  is the union of the edges it crosses (viewed as embedded in  $\mathbb{C}$ ). We say random walk from  $v$  does a **full turn** in  $A(x, r, R)$ , if the random walk trajectory intersects every simple curve starting from  $\{|z| = r\}$  and ending in  $\{|z| = R\}$  before exiting  $A(x, r, R)$ . We will write  $X[a, b]$  for the random walk trajectory between times  $a$  and  $b$ . We will allow ourselves to see  $X[a, b]$  and the loop-erased walk  $Y$  either as a sequences or as sets depending on the place but this should not lead to any confusion. For any continuous curve  $\lambda \in \mathbb{C}$ , with a

slight abuse of terminology we will freely say that “ $(X_t, t \geq 0)$  **crosses (or hits)**  $\lambda$  at time  $t > 0$ ” to mean that  $X_t \neq X_{t-}$  and the edge  $[X_{t-}, X_t]$  intersects the range of the curve  $\lambda$ .

In this section and the next, we will always assume that the loop-erased walk is generated by erasing loops chronologically from a simple random walk. We will allow ourselves to refer to the simple random walk associated to a loop-erased walk without further mention of this.

**Lemma 4.3.** *Fix  $0 < r < R$ ,  $x \in \mathbb{C}$  and  $\Delta = R - r$ . There exists a constant  $C > 0$  depending only on  $R/r$  such that if  $\delta \leq Cr\delta_0$  where  $\delta_0$  is as in item [v](#), and there exists a constant  $\alpha = \alpha(R/r) > 0$  such that for all  $v \in A(x, r + \Delta/3, R - \Delta/3)^{\# \delta}$ , the probability that the random walk starting at  $v$  does a full turn before exiting  $A(x, r, R)$  is at least  $\alpha$ .*

*Proof.* We use the uniform crossing assumption here and use the notations and terminology as described in Section [4.1](#). It is easy to see that we can find a sequence of rectangles  $R_1, R_2, \dots, R_k \subset A(x, r + \Delta/4, R - \Delta/4)^{\# \delta}$  where each such rectangle is a rectangle of the form  $\varepsilon R + z$  or  $\varepsilon R' + z$  (i.e. a scaling and translation of  $R$  or  $R'$ ) such that the starting ball of  $R_i$  coincides with the target ball of  $R_{i-1}$ ,  $v$  is in the starting ball of  $R_1$  and the following holds. If the simple random walk iteratively moves from the starting ball to the target ball of  $R_i$  for each  $i = 1, \dots, k$  such that the starting vertex of  $R_{i+1}$  is the vertex where the walk enters the target ball of  $R_i$ , then the walker accomplishes a full turn in  $A(x, r, R)$ . Here we can choose the scaling  $\varepsilon$  as a function of the ratio  $r, R$  and the number  $k$  to be bounded above by a constant  $k_0(R/r)$ . Applying the uniform crossing estimate and the Markov property of the walk, we see that this probability is bounded below by  $\alpha_0^{k_0}$ , thus completing the proof.  $\square$

Actually we will need estimates such as Lemma [4.3](#) to hold even when we condition on the exit point of the annulus which we will prove now. The first step is to prove a conditional version of the uniform crossing estimate.

**Lemma 4.4.** *Fix  $0 < r < R$  and  $\epsilon < (R - r)/3$ . There exists a constant  $C$  depending only on  $R/r$  and  $\epsilon/r$  and there exists  $\alpha = \alpha(R/r, \epsilon/r) > 0$  such that if  $\delta \leq Cr\delta_0$  and  $x \in \mathbb{C}$ , the following holds. Let  $\tau$  be the stopping time where the random walk exits  $A(x, r, R)^{\# \delta}$ . Let  $R$  be a rectangle of the form  $y + [0, 3\eta] \times [0, \eta]$  such that  $R \subset A(x, r + \epsilon, R - \epsilon)$  and  $\eta \geq \epsilon$ . Let  $B_1$  and  $B_2$  be balls defined as in the uniform crossing estimate, i.e.  $B_1 = y + B((\frac{\eta}{2}, \frac{\eta}{2}), \frac{\eta}{4})$  and  $B_2 = y + B((\frac{5\eta}{2}, \frac{\eta}{2}), \frac{\eta}{4})$ . For all  $v \in B_1^{\# \delta}$  and  $u \in \partial A(x, r, R)^{\# \delta}$  such that  $\mathbb{P}_v[X_\tau = u] > 0$ ,*

$$\mathbb{P}_v[X \text{ hits } B_2 \text{ before exiting } R | X_\tau = u] > \alpha.$$

*Proof.* Let  $h(v) = \mathbb{P}_v[X_\tau = u]$ . We start by giving a rough bound on  $h$  on  $A(x, r + \epsilon, R - \epsilon)$ . Let us fix  $v, v' \in A(r + \epsilon, R - \epsilon)$ . Since  $h$  is harmonic, there exists a path  $\gamma = \{v, v_1, \dots\}$  from  $v$  to  $\partial A^{\# \delta}(x, r, R)$  along which  $h$  is nondecreasing. Also since  $h$  is harmonic and bounded, if  $\tau_\gamma$  denotes the hitting time of  $\gamma \cup \partial A(x, r, R)$  we have

$$h(v') = \mathbb{E}_{v'} h(X_{\tau_\gamma}) \geq h(v) \mathbb{P}_{v'}[X_{\tau_\gamma} \in \gamma].$$

Using the crossing estimate a bounded number of times as in the proof of Lemma [4.3](#), it is clear that there exists a constant  $\beta = \beta(R/r, \epsilon/r)$  independent of  $\delta$  and  $v'$  such that

$$\mathbb{P}_{v'}[X \text{ does a full turn in } A(x, r, r + \epsilon) \text{ and in } A(x, R - \epsilon, R) \text{ before exiting } A(x, r, R)] \geq \beta.$$

We see that on the above event we have  $X_{\tau_\gamma} \in \gamma$  so we have proved

$$\forall v, v' \in A(x, r + \epsilon, R - \epsilon)^{\# \delta}, \beta h(v) \leq h(v') \leq 1/\beta h(v).$$

Now together with the Markov property and the uniform crossing estimate this gives

$$\mathbb{P}_v[X \text{ hits } B_2 \text{ before exiting } R \text{ and } X_\tau = u] \geq \mathbb{P}_v(X \text{ hits } B_2 \text{ before exiting } R) \inf_{v' \in B_2^{\# \delta}} h(v') \geq \alpha \beta h(v).$$

Dividing by  $h(v)$ , the proof is complete.  $\square$

Using a bounded number of rectangles to surround the center  $x$  in  $A(x, r, R)$  as in the above lemma, we get the following corollaries

**Lemma 4.5.** *Suppose we are in the setup of Lemma 4.3 so that  $r, R, C, \delta_0$  are as in Lemma 4.3. Let  $\tau$  be the stopping time when the random walk exits  $A(x, r, R)^{\# \delta}$ . Let  $u \in \partial A(x, r, R)^{\# \delta}$  such that  $\mathbb{P}_v(X_\tau = u) > 0$ . Then for all  $\delta \in (0, Cr\delta_0)$  conditioned on  $X_\tau = u$ , the  $\mathbb{P}_v$  probability of the random walk doing a full turn in  $A(x, r, R)$  is at least  $\alpha = \alpha(R/r) > 0$ .*

**Lemma 4.6.** *Fix  $0 < r < R$ . There exists a constant  $C = C(R/r) > 0$  and  $\alpha(R/r)$  such that for  $\delta \leq Cr\delta_0$  and for all  $x, v \in A(x, r + \frac{R-r}{3}, R - \frac{R-r}{3})^{\# \delta}$  and  $u$  such that  $\mathbb{P}_v(X_\tau = u) > 0$ , where  $\tau$  is the exit time of  $B(x, R)$ , we have*

$$\mathbb{P}_v[X \text{ enters } B(x, r) \text{ before exiting } B(x, R) | X_\tau = u] > \alpha.$$

The next lemma establishes an exponential tail for the winding of the simple random walk in an annulus conditioned to exit at a vertex. This estimate is needed to establish uniform integrability for the winding of the loop erased walk. Recall that we write  $X[t, t']$  for the random walk path between times  $t$  and  $t'$  and  $W(\gamma, x)$  for the topological winding of a path  $\gamma$  around  $x$ .

**Lemma 4.7.** *Fix  $r, R, \delta_0, C$  as in Lemma 4.3. There exists  $\alpha = \alpha(R/r) > 0$  and  $c$  such that for all  $x \in \mathbb{C}$ ,  $\delta \in (0, Cr\delta_0)$ ,  $v \in A(x, r + \frac{R-r}{3}, R - \frac{R-r}{3})^{\# \delta}$ ,  $u$  such that  $\mathbb{P}_v(X_\tau = u) > 0$  where  $\tau$  is the exit time of  $A(x, r, R)$  and  $n \geq 1$ , we have*

$$\mathbb{P}_v \left[ \sup_{\mathcal{Y} \subset X[0, \tau]} |W(\mathcal{Y}, x)| \geq n | X_\tau = u \right] \leq c\alpha^n.$$

where the supremum is over all continuous paths  $\mathcal{Y}$  obtained by erasing portions from  $X[0, \tau]$ .

*Proof* The proof is divided in three steps. First we construct a set  $U \subset A(x, r, R)$  such that if the random walk is started in  $U$  it has a positive probability to hit  $u$  without winding around  $x$ . In the second step we use the conditional crossing estimate (Lemma 4.4) to show that with positive probability the random walk started from the bulk of the annulus  $A(x, r, R)$  can hit  $U$  without winding either. The basic idea is then to apply the Markov property. However the conditional crossing estimate requires the source and target points to be far from the boundary of the annulus. It turns out that points which are near the boundary component of the annulus containing  $u$  do not cause any problem in this estimate. However, starting the random walk from near the opposite boundary component requires an additional trick which constitutes the third step of the proof (essentially this amounts to shrinking the annulus).

**Step 1.** Let us assume first that  $u$  is a point on the outer boundary of the annulus. Up to a rotation and a translation of the graph, we can assume that  $x = 0$  and  $u$  is on the negative real axis. We also simplify notations by writing  $A = A(x, r, R)$ . For a continuous path  $\gamma$  we write  $\gamma^{\# \delta}$  for a discrete path staying at distance  $\delta/\delta_0$  of  $\gamma$ .

We start as in the previous proof by controlling the function  $h(v) = \mathbb{P}_v[X_\tau = u]$ . More precisely we claim that there exists  $c$  such that for all  $\rho \leq \frac{R-r}{10}$ , for all  $\delta$  small enough (depending on  $\rho$ ), for all  $a \in (A \cap \partial B(u, \rho))^{\# \delta}$  and  $b \in (\{z \mid \arg(z - u) \in [-\frac{\pi}{4}, \frac{\pi}{4}]\} \cap \partial B(u, \rho))^{\# \delta}$ ,

$$h(b) \geq ch(a). \quad (4.2)$$

Let  $b \in (\{z \mid \arg(z - u) \in [-\frac{\pi}{4}, \frac{\pi}{4}]\} \cap \partial B(u, \rho))^{\# \delta}$  be fixed and let  $a$  be a point in  $(A \cap \partial B(u, \rho))^{\# \delta}$ . Since  $h$  is harmonic there exists a path  $\gamma$  from  $a$  to  $u$  along which  $h$  is non-decreasing and we can assume that  $\gamma$  does not intersect  $(A \cap \partial B(u, \rho))^{\# \delta}$  outside of  $a$  (otherwise change  $a$  to be the last intersection point). Let  $\tau_\gamma$  be the first hitting time of  $\gamma \cup \partial A$ . By harmonicity we have

$$h(b) = \mathbb{E}_b[h(X_{\tau_\gamma})].$$

On the other hand, by the crossing estimate, the random walk has a positive probability  $c$  independent of  $\rho$  to hit  $\gamma$  irrespective of the relative positions of  $a, b, u$ . (For example this is at least the probability to surround  $\partial B(u, \rho) \cap A$  in the clockwise and anticlockwise directions, staying in the annulus  $A(u, 99\rho/100, 101\rho/100)$ .) Hence using this event to lower bound the expectation, we find  $h(b) = \mathbb{E}_b[h(X_{\tau_\gamma})] \geq h(a)c$ . This proves the claim (4.2).

A second claim is that  $h$  increases at least polynomially if the starting point is moved closer to  $u$ . More precisely, there exists  $\alpha$  such that, for all  $\rho \leq \frac{R-r}{10}$ , for all  $n$ , for all  $\delta$  small enough depending on  $n$  and  $\rho$ ,

$$\sup_{A \cap \partial B(u, \rho)^{\# \delta}} h \leq \alpha^n \sup_{A \cap \partial B(u, 2^{-n}\rho)^{\# \delta}} h. \quad (4.3)$$

Indeed, by the crossing estimate, there is a uniform *upper* bound on the probability that the random walk started on a circle of radius  $\rho$  reaches a circle of radius  $\rho/2$  before hitting the boundary. Then using the same bound iteratively proves (4.3).

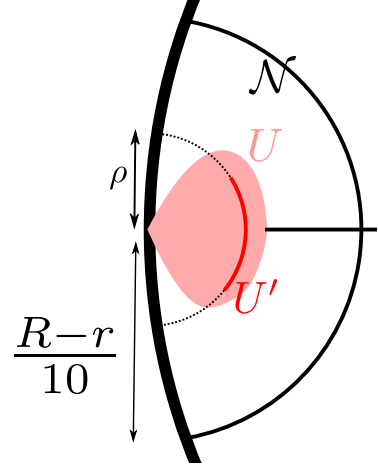
Using (4.2) and (4.3), we can define  $\mathcal{N} = A^{\# \delta} \cap B(u, \frac{R-r}{10})$  and  $U' = (\{z \mid \arg(z - u) \in [-\frac{\pi}{4}, \frac{\pi}{4}]\} \cap \partial B(u, \rho))^{\# \delta}$  with  $\rho$  small enough such that

$$\forall v \in U', h(v) \geq 2 \sup_{\partial \mathcal{N} \cap A^{\# \delta}} h.$$

Now let us define  $U = \{v \mid h(v) \geq \inf_{U'} h\}$ . The same trick of considering non-decreasing path shows that  $U$  is connected. It is also obvious that  $u \in U$ ,  $U' \subset U$  and  $U \subset \mathcal{N}$ .

This easily concludes the first step. Indeed

$$\mathbb{P}_v[X \text{ exits } \mathcal{N} \text{ before } \tau \mid X_\tau = u] \leq \frac{\sup_{\partial \mathcal{N} \cap A^{\# \delta}} h}{h(v)} \leq \frac{\inf_{U'} h}{2h(v)} \leq \frac{1}{2} \quad (4.4)$$



**Figure 9:** Illustration of the sets used in Step 1 of Lemma 4.7. The heavy black curve on the left is the outer boundary of the annulus.

Figure 1: Schematic diagram of the geometry of the system. A large outer circle of radius  $R$  is centered at the origin. A smaller inner circle of radius  $r$  is also centered at the origin. A dashed circle of radius  $\frac{R-r}{10}$  is concentric with the inner circle. A red shaded region  $U'$  is located on the left side of the inner circle, with a radius  $\rho$ . The distance from the center of the inner circle to the center of  $U'$  is labeled  $\ell_+$ . The distance from the center of the outer circle to the center of  $U'$  is labeled  $\ell_-$ . The distance from the center of the outer circle to the center of the dashed circle is labeled  $\frac{R-r}{10}$ .

**Figure 10:** The global picture of the annulus in the proof of Lemma 4.7.

Using Lemma 4.4 to see that the random walk has a positive probability to travel in a straight line along  $\ell_-$  shows that we can start at any point in  $\ell_-$ , i.e

As a consequence by the Markov property, we can find  $\alpha$  and  $C$  such that

**Step 3.** Now let  $v$  be a point in  $A(r + \frac{R-r}{5}, R - \frac{R-r}{5})$ . Let  $\tilde{\tau}$  be the hitting time of  $B(x, r + \frac{R-r}{10})$ . By Lemma 4.4, uniformly over  $v$ , the random walk conditioned on  $X_\tau = u$  has a positive probability to hit  $U'$  and therefore  $U$  before  $\tilde{\tau}$ . Therefore by eq. (4.4) we can find  $\beta$  such that

for all  $v \in A(r + \frac{R-r}{5}, R - \frac{R-r}{5})$ . Using this estimate together with eq. (4.5), we get

However notice that for any path  $\gamma$  in the smaller annulus  $A(r + \frac{R-r}{10}, R)$ , we have the deterministic bound  $\sup_{\mathcal{Y}} |W(\mathcal{Y}, x)| \leq 2\pi(I_+ + 1)$ . In particular replacing in the last equation we obtain

for all  $v \in A(r + \frac{R-r}{5}, R - \frac{R-r}{5})$ . Let  $r' = r + (R-r)/10$  and  $R' = R$ . Let  $\tau'$  be the exit time from  $A(r', R')$ . Then since  $r + \frac{R-r}{5} \leq r' + \frac{2}{5}(R-r) = r' + \frac{R'-r'}{3}$  we deduce that

for all  $v \in A(r' + \frac{R'-r'}{3}, R' - \frac{R'-r'}{3})$ . This is exactly the statement of the lemma, except with  $r'$  and  $R'$  instead of  $r$  and  $R$ .



Finally it is clear that the above proof extends to the case where  $u$  is a point in the inner boundary instead of being in the outer boundary.  $\square$

Recall that we decompose the winding into  $h^{\#\delta} = h_t^{\#\delta} + \varepsilon^{\#\delta}$  (see eq. (1.3)). We have already dealt with the limit of  $h_t^{\#\delta}$  as  $\delta \rightarrow 0$  in the continuum part. It now remains to say that  $\varepsilon^{\#\delta}$  does not contribute because when  $x \neq x'$  and  $t$  is large,  $\varepsilon^{\#\delta}(x)$  and  $\varepsilon^{\#\delta}(x')$  are nearly independent. This is proved in Section 4.4 by constructing a coupling of the sub tree around  $x$  and  $x'$  with independent variables. This coupling is built by sampling tree branches in the right order using Wilson algorithm and analysing carefully which part of the graph the random walks visits while performing the algorithm. In particular, a crucial step is to control the probability that the loop-erased walk from  $x$  comes close to  $x'$  which we now prove.

**Proposition 4.8.** *Let  $D \subset \mathbb{C}$  be a domain and let  $u, v \in D$ . Let  $r = |u - v| \wedge \text{dist}(v, \partial D) \wedge \text{dist}(u, \partial D)$ . Let  $v^{\#\delta}$  be the closest vertices to  $v$  in  $G^{\#\delta}$ . Let  $\gamma$  be a loop erased walk starting from  $v^{\#\delta}$  until it exits  $D^{\#\delta}$ . Then with  $\delta_0$  as in the uniform crossing assumption item  $v$  in Section 4.1, for all  $\delta \in (0, C\delta_0]$  for some universal constant  $C > 0$ , for all  $n \in (0, \log_4(Cr\delta_0/\delta - 1)) \cap \mathbb{N}$ ,*

$$\mathbb{P}(|\gamma - u| < 4^{-n}r) < \alpha^n$$

for some universal constant  $\alpha \in (0, 1)$ .

*Proof.* We assume  $|u - v| = r$  for otherwise we can wait until the simple random walk comes closer to  $u$ . The idea for the proof is the following. If the loop erased walk comes within distance  $4^{-k}r$  to  $u$ , then after the *last* time the simple walk was within distance  $4^{-k}r$  to  $u$ , it crossed  $k$  annuli without performing a full turn. The probability of this event is exponentially small in  $k$  via Lemma 4.5.

Now we write the details. Let  $i_{\max}(\delta) = \lfloor \log_4(C\delta_0 r/\delta) \rfloor$ . Let  $\{C_i\}_{0 \leq i \leq i_{\max}}$  denote the circle of radius  $4^{-i}r$  around  $u$  and define  $C_{-1} = \partial D$ . We inductively define a sequence of times  $\{\tau_k\}_{k \geq 0}$  as follows. We have  $\tau_0 = 0$ . Having defined  $\tau_k$  to be a time when the random walk crosses (or hits) some circle  $C_{i(k)}$ , we define  $\tau_{k+1}$  to be the smallest time when leaving the annulus defined by  $C_{i(k)-1}$  and  $C_{i(k)+1}$ , and define  $i(k+1)$  to be the index of the circle by which the random walk leaves the annulus. If  $i(k) = i_{\max}$ , we define  $\tau_{k+1}$  to be smallest time after  $\tau_k$  when the walk leaves the ball defined by  $C_{i_{\max}-1}$ . We stop if we leave  $D$  and let  $N$  be the largest index of  $\tau$  after which we stop.

Let  $\mathcal{S} := (X_{\tau_k})_{0 \leq k \leq N}$  denote the sequence of crossing positions of the  $C_i$ . Notice that conditioned on any  $\mathcal{S}$ , if  $k < N$ , the simple random walk between  $X_{\tau_k}$  and  $X_{\tau_{k+1}}$  is a simple random walk in the annulus  $A(u, C_{i(k)-1}, C_{i(k)+1})$  conditioned to exit at  $X_{\tau_{k+1}}$ . Furthermore by the Markov property of the walk, conditioned on  $\mathcal{S}$ , the portions of random walk  $(X[\tau_k, \tau_{k+1}])_k$  are independent.

On the event  $|\gamma - u| < 4^{-n}r$ , the sequence of positions  $\mathcal{S} := \{X_{\tau_k}\}_{0 \leq k \leq N}$  contains an index when the random walk crosses  $C_n$  since the loop-erased walk is obviously a subset of the walk. Let  $\kappa$  be the index of the last crossing of  $C_n$  by the walk and let  $\gamma'$  be the path obtained by erasing loops from  $X[0, \tau_\kappa]$ , i.e the “current loop-erased path” at time  $\tau_\kappa$ . On the event  $|\gamma - u| < 4^{-n}r$ , necessarily the random walk did not hit  $\gamma'$  after  $\tau_\kappa$ , otherwise the part of the path closer to  $u$  would have been erased. In particular the random walk did not do any full turn after time  $\tau_\kappa$ . Also by construction we have  $N - \kappa \geq n$ .

Therefore we see that conditioned on the sequence of position  $\mathcal{S}$ , the event  $|\gamma - u| < 4^{-n}r$  is included in the event that there was no full turn in the last  $n$  intervals  $[\tau_k, \tau_{k+1}]$ . Choosing the constant  $C$  to be the constant from Lemma 4.5 associated with annuli of aspect ratio  $R/r = 16$ , we can apply that lemma for each annulus, since for each  $i \leq i_{\max}$ , if  $r_i$  is the radius of  $C_i$ , we have

$\delta \leq Cr_i\delta_0$  by our choice of  $i_{\max}(\delta)$ . Hence using the independence noted above, the conditional probability of this is at most  $\alpha^n$  for some  $\alpha < 1$ , which concludes the proof.  $\square$

### 4.3 Tail estimate for winding of loop-erased random walk

Recall that for any vertex  $v \in D^{\#\delta}$ , Let  $\gamma_v^{\#\delta}$  denote the branch of the wired UST in  $D^{\#\delta}$  from  $v$  to the wired vertex and we parametrise it via its capacity minus  $\log R(v, D)$  in the domain  $D$ . We wish to take limits of moments of  $W(\gamma_v[0, t]^{\#\delta}, v)$  for a fixed  $t$  as  $\delta \rightarrow 0$  and hence we need to estimate the behaviour of the winding in the discrete as  $\delta$  varies. We prove that in fact  $W(\gamma_v[0, t]^{\#\delta}, v)$  has (stretched) exponential tail uniformly in  $\delta$ .

We will make use of the following elementary observation. Say that a random variable  $X$  has stretched exponential tail (with parameters  $\alpha, c$ ) if  $\mathbb{P}(|X| \geq t) \leq c^{-1} \exp(-ct^\alpha)$  for all  $t > 0$ . The proof of the following elementary lemma is left to the reader.

**Lemma 4.9.** *Suppose  $(X_i)_{i \geq 1}$  and  $N$  are random variables with stretched exponential tail with parameters  $\alpha$  and  $c$ . Then  $\sum_{i=1}^N X_i$  has stretched exponential tail.*

**Proposition 4.10.** *Let  $\text{Diam}(D)$  denote the diameter of  $D$  and let  $v \in D^{\#\delta}$ . Fix  $t > -10 \log(\text{Diam}(D) \wedge |v - \partial D|)$ . Then there exists a universal constants  $C, c > 0$  and  $\alpha$  such that for all  $\delta < Ce^{-t}\delta_0$  and  $n \geq 1$ ,*

$$\mathbb{P}\left(\sup_{t \leq t_1, t_2 \leq t+1} |W(\gamma_v[0, t_1]^{\#\delta}, v) - W(\gamma_v[0, t_2]^{\#\delta}, v)| > n\right) < Ce^{-cn^\alpha}.$$

for some  $\alpha \in (0, 1]$ .

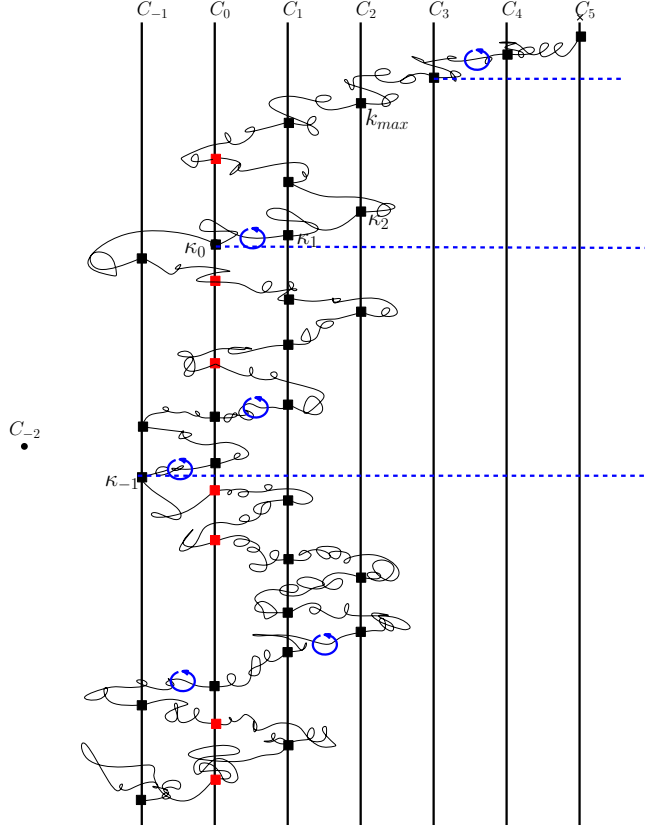
We first set up a few notations. which are completely analogous to the ones in the proof of Proposition 4.8 except that here we are considering circles of growing size.

Let  $r_i = (4e)^{i-1}e^{-t}$  for  $i \geq -1$  and  $r_{-2} = 0$ . Let  $C_i$  be the circle of radius  $r_i$  centered at  $v$  as long as  $C_i \subset D$ . As soon as  $C_i$  is not a subset of  $D$ , define  $C_i = \partial D$  (call the maximum index  $i_{\max}$ ). Let  $X$  be a random walk from  $v$  run until it leaves the domain  $D$ . Let  $Y(t)$  be the loop-erasure of  $X$ , reparametrised by capacity seen from  $v$  (so in particular  $Y(0) \in \partial D^{\#\delta}$  and  $Y(\infty) = v$ ). We emphasise that we are indexing circles starting from  $i = -2$ .

We inductively define a sequence of times  $\{\tau_k\}_{k \geq 0}$  as follows. We have  $\tau_0 = 0$  and  $i(0) = -2$ ,  $\tau_1$  is the time the random walk crosses  $C_{-1}$  and  $i(1) = -1$ . Having defined  $\tau_k$  to be the smallest time when the random walk crosses (or hits) some circle  $C_{i(k)}$ , we define  $\tau_{k+1}$  to be the smallest time when it hits  $C_{i(k)-1}$  or  $C_{i(k)+1}$  and define  $i(k+1)$  to be the index of the circle it crosses. When  $i(k) = -1$  we define  $\tau_{k+1}$  only as the next crossing of  $C_0$ . We define  $\mathcal{S} = (X_{\tau_k})_{k \geq 0}$  the sequence of crossing positions. For any  $j$ , we also define  $\mathcal{V}_j$  to be the sequence of crossings of the circle  $C_j$ , i.e  $\mathcal{V}_j = \{k | i(k) = j\}$ . We will call sets of the form  $X[\tau_k, \tau_{k+1}]$  **elementary piece of random walk**. In this proof we also only call **crossing** the times of the form  $\tau_k$ .

A first observation from Koebe's 1/4 theorem is that we only need to look at the portion of  $Y$  from the time it first enters  $C_1$  until it first enters  $C_0$ . Let  $\mathcal{Y}$  denote this set. The main idea of the proof will be to show that that this portion of the loop erased walk can be generated by erasing loops from a small number of elementary pieces of random walk. Then Lemma 4.7 will show that each piece does not contribute too much to the winding.

To control the number of pieces, we will use two elements. First conditionally on  $\mathcal{S}$  we will argue that we only need to look at the last few visits of the simple random walk to any circle because everything else was erased by a loop. Secondly we will show that the sequence  $\mathcal{S}$  is not too



**Figure 11:** Schematic illustration of the proof of Proposition 4.10. Time runs upward. Blue circular arrows indicate a full turn in an annulus. The times  $\tau_k$  are indicated with a box (red boxes correspond to the times  $\tau_k^j$  for  $j = 0$  in Lemma 4.12). Horizontal dashed blue lines correspond to the last time  $\kappa_j$  that there is a full turn. In this example  $I = 2$ .

badly behaved even when we are looking close to the last visit to  $C_0$ . Note that this is non trivial because the last visit to  $C_0$  is very far from being a stopping time. We now proceed to the actual proof, writing each of these steps as lemmas.

We first note some deterministic facts about the loop erasure (Lemma 4.11), therefore until further notice we work on a given realisation of the random walk. Let  $k_{\max} - 1$  be the index of the last crossing of  $C_1$ , and let  $N$  be the first  $k$  such that  $i(k) = i_{\max}$ .

Let  $\kappa_{-1}$  be the last  $k$  such that  $i(k) = -1$  and in the interval  $[\tau_k, \tau_{k+1}]$  the random walk did a full turn in  $A(v, r_{-1}, r_0)$ . If there was no such full turn, set  $\kappa_{-1} = 0$ . Now we define  $\kappa_i$  inductively as follows. If there was a full turn in the annuli  $A(v, r_i, r_{i+1})$  after time  $\kappa_{i-1}$  but before  $k_{\max}$ , then  $\kappa_i$  is the index of the interval where the last of these full turns occurred. Otherwise  $\kappa_i$  is the index of the first crossing of  $C_i$  after  $\kappa_{i-1}$  but before  $k_{\max}$ . If there is no crossing of  $C_i$  between  $\kappa_{i-1}$  and  $k_{\max}$  define  $\kappa_i = +\infty$ . Finally we define  $I = \max\{i | \kappa_i < \infty\}$  and for every  $i \leq I$ , we let  $\mathcal{G}_i$  be the set of visits of  $C_i$  after  $\kappa_{i-1}$  but before  $k_{\max}$ , i.e  $\mathcal{G}_i = \{k \in \mathcal{V}_i | \kappa_{i-1} \leq k \leq k_{\max}\}$ .

**Lemma 4.11.** *The portion  $\mathcal{Y}$  of  $Y$  from the first time it enters  $C_1$  until the first time it enters  $C_0$ ,*

is a subset of

$$\bigcup_{0 \leq i \leq I} \bigcup_{k \in \mathcal{G}_i} X[\tau_k, \tau_{k+1}].$$

Furthermore, one can write  $\mathcal{Y} = \bigcup_{k \in \mathcal{G}_i} \mathcal{Y}_k$  where  $\mathcal{Y}_k$  are disjoint intervals of the loop erased random walk of the form  $(Y_{j_k}, Y_{j_k+1}, \dots, Y_{j_k+i_k})$  and  $\mathcal{Y}_k \subset X[\tau_k, \tau_{k+1}]$ .

*Proof.* We consider the chronological erasure of loops so at each time  $T$  we have a loop-erased path  $Y^{(T)}$  obtained from erasing loops for the random walk  $(X_t)_{t \leq T}$ .

Notice that by construction, after time  $\tau_{k_{\max}}$  the random walk does not return to  $B(0, r_1)$  so  $\mathcal{Y} \subset Y^{(\tau_{k_{\max}})}$  (it might be smaller because more loops can occur, erasing further the final set  $\mathcal{Y}$ , but no more points can be added to  $\mathcal{Y}$  since we do not return to the annulus). This justifies looking only at  $k \leq k_{\max}$  above.

The introduction of the  $\kappa_i$  is justified by the following topological fact: if  $X$  does a full turn between times  $\tau_k$  and  $\tau_{k+1}$ , then  $Y^{(\tau_{k+1})} \subset B(v, r_{i(k)+1})$ . Indeed if  $Y^{(\tau_k)} \subset B(v, r_{i(k)+1})$  the statement is trivial. Otherwise  $Y^{(\tau_k)}$  includes a path from  $C_{i(k)-1}$  to  $C_{i(k)+1}$  which has to be crossed by the full turn at some time  $T$  and this erases everything outside of  $B(v, r_{i(k)+1})$ .

In particular, we see that  $Y^{\tau_{\kappa-1}+1} \subset B(v, r_0)$ . This implies that  $\mathcal{Y} \subset X[\tau_{\kappa-1}+1, \tau_{k_{\max}}]$  since all the rest of  $X$  either was erased or contributes to parts of  $Y$  outside of  $\mathcal{Y}$ . Using the same argument, we can erase the part of the walk before  $\tau_{\kappa_0}$  which is outside of  $C_1$ . But we need to keep all the visits to  $C_0$  (that is all the elementary pieces of random walk in  $\mathcal{G}_0$ ). So we get

$$\mathcal{Y} \subset \bigcup_{k \in \mathcal{G}_0} X[\tau_k, \tau_{k+1}] \cup X[\tau_{\kappa_0+1}, \tau_{k_{\max}}].$$

We now proceed by induction to complete the proof of the first part of the statement.

For the last sentence, note that the edges of  $Y$  are created by the random walk in order from the tip to the boundary. Therefore the set of edges created in an interval of time by the random walk has to form an interval in the loop-erased walk  $Y$ . We emphasise that here we only look at the times when edges are added to  $Y$ , the full loop-erased walk, and not the loop-erased path  $Y^{(t)}$  at any intermediate point of time.  $\square$

The next step is to control the law of the size of the sets  $\mathcal{G}_j$ , therefore we go back to considering  $X$  as random.

**Lemma 4.12.** *There exists  $c > 0$ ,  $\alpha \in (0, 1)$  and  $C > 0$  such that, for all  $\delta \leq ce^{-t}\delta_0$ , for all  $n$ , for all  $j \geq 1$ ,*

$$\mathbb{P}(I \geq n) \leq C\alpha^n, \quad \mathbb{P}(|\mathcal{G}_j| \geq n) \leq C^{-1} \exp(-Cn^\alpha).$$

As a consequence,  $\sum_{1 \leq j \leq I} |\mathcal{G}_j|$  has stretched exponential tail.

*Proof.* The first statement is fairly straightforward. Condition on the sequence of crossing positions  $\mathcal{S} = (X_{\tau_k})_{k \geq 0}$ . Recall that by construction  $i(k_{\max} - 1) = 1$ , therefore for every  $i \geq 2$ , just after last crossing  $\tau_k$  of  $C_i$  before  $k_{\max}$  the random walk goes to  $C_{i-1}$ . In particular if there is a full turn within the annulus  $A(v, r_i, r_{i+1})$  during  $[\tau_k, \tau_{k+1}]$ , then by definition  $\kappa_{i+1} = +\infty$ . By Lemma 4.5 the conditional probability of a full turn has a uniformly positive probability to happen for each  $i$  so we are done.

For the second statement we have to work on the law of  $\mathcal{S}$  so it is more tricky. Note that by Lemma 4.5, we immediately see that the number of visits to  $C_i$  after  $\tau_{\kappa_i}$  has geometric tail. (Indeed,

conditionally on  $\mathcal{S}$ , on the event that there are  $n$  visits to  $C_i$  after  $\tau_{\kappa_i}$  is event that the last  $n$  visits to  $C_i$  – which are measurable with respect to  $\mathcal{S}$  – were not followed by a full turn in the annulus  $A(v, r_i, r_{i+1})$ . But a full turn in this annulus has uniformly positive probability by Lemma 4.5 at each visit, conditionally on  $\mathcal{S}$ . So this conditional probability has geometric tail, and thus so does the unconditional probability).

However  $|\mathcal{G}_i|$  is the number of visits to  $C_i$  after  $\kappa_{i-1}$ , so we also need to exclude the possibility that the random walk alternates many times between  $C_i$  and  $C_{i+1}$  without crossing  $C_{i-1}$ . We are actually going to prove that the number of crossings of  $C_i$  between two successive crossings of  $C_{i-1}$  has an exponential tail. As in the proof of Proposition 4.8, the difficulty comes from the fact that we are looking close to random times which are not stopping times, so we don't know the law of the walk. To get around this issue we will need to be more careful and discover the set  $\mathcal{S}$  in steps.

For any  $j$ , let us define  $\tau^j$ ,  $i^j$  and  $\mathcal{S}^j$  as before but using  $C_j$  as the reference smallest circle rather than  $C_{-1}$  as above. More precisely, we define  $\tau_0^j = 0$ ,  $\tau_1^j$  to be the first crossing of  $C_j$ . Then by induction, whenever  $i^j(k) = j$  we define  $\tau_{k+1}^j$  to be the first crossing of  $C_{j+1}$  after  $\tau_k^j$  and otherwise if  $i^j(k) > j$  then we set  $\tau_{k+1}^j$  to be the first crossing of  $C_{i^j(k) \pm 1}$  after  $\tau_k^j$ . Finally we set  $\mathcal{S}^j = (X(\tau_k^j))_k$ . Let us modify this a bit and define a sequence  $\tilde{\mathcal{S}}^j$  by inserting in the sequence  $\mathcal{S}^j$  the point  $X(\tau_{k_{\max}-1})$  where the last crossing of  $C_1$  happens. Let  $\{\tilde{\tau}_k^j\}_{k \geq 0}$  be the times corresponding to the sequence  $\tilde{\mathcal{S}}^j$ .

As before, conditionally on  $\tilde{\mathcal{S}}^j$  the pieces of random walk  $X[\tilde{\tau}_k^j, \tilde{\tau}_{k+1}^j]$  are independent. For any  $k$  such that  $i^j(k) = j$ , the piece of random walk  $X[\tilde{\tau}_k^j, \tilde{\tau}_{k+1}^j]$  is distributed as a random walk starting at some point on  $C_j$  and conditioned on its exit point of  $B(v, r_{j+1})$ . By Lemma 4.6, this random walk has a strictly positive probability to intersect  $C_{j-1}$  and this probability is independent of everything else. Therefore in  $\tilde{\mathcal{S}}^j$ , looking at the successive visits to  $C_j$ , the gaps between visits where the walk also crossed  $C_{j-1}$  have a uniform exponential tail, even if we look at positions relative to  $\tau_{k_{\max}}$ . Going back to  $\mathcal{S}$ , this translates immediately in the fact that the number of crossings of  $C_j$  between two successive crossings of  $C_{j-1}$  has an exponential tail.

Now let  $\mathcal{I}_{j-1} = \{k \in \mathcal{V}_{j-1} : k \geq \kappa_{j-1}\}$  (which has geometric tail by the initial observation) and for  $k \in \mathcal{I}_{j-1}$ , let  $\sigma(k)$  be the successor of  $k$ , i.e., the next smallest element in  $\mathcal{I}_{j-1}$ . Then

$$|\mathcal{G}_j| \leq \sum_{k \in \mathcal{I}_{j-1}} \#\{\text{visits to } C_j \text{ between } k \text{ and } \sigma(k)\}.$$

By the above discussion, conditionally on  $\tilde{\mathcal{S}}^j$ , each of the terms in the above sum has geometric tail. Moreover the number of terms has geometric tails. By Lemma 4.9, it follows that  $|\mathcal{G}_j|$  has stretched exponential tail. The lemma follows.  $\square$

Now it is easy to complete the proof of Proposition 4.10 using Lemma 4.7. By Lemma 4.11 we can write  $\mathcal{Y} = \bigcup_k Y_k$  with for all  $k$ ,  $Y_k \subset X[\tau_k, \tau_{k+1}]$ . Therefore the winding around  $v$  of any  $Y_k$  is bounded by the maximal winding difference between two times in  $[\tau_k, \tau_{k+1}]$  which has exponential tail by Lemma 4.7. By Lemma 4.12 the number of terms in the union has stretched exponential tail, so the proposition follows from applying Lemma 4.9.  $\square$

**Remark 4.13.** In fact, with relatively little extra work we could obtain exponential tails in Proposition 4.10 but this is not necessary for us.

**A generalisation.** We now record a deduction of the proof of Proposition 4.10 which will be useful in our upcoming paper of the extension of our universality result for graphs embedded in a general Riemann surface [5]. A crucial difference of the setup we consider here compared to Proposition 4.10 is that now we don't stop the walk when it reaches the boundary of a finite domain. Instead we stop at the  $j$ th crossing of a circle for any arbitrarily large  $j$ . Further, we only assume the validity of the crossing assumption inside a small but macroscopic ball.

Let us describe the full setup. Let  $G^{\#\delta}$  be a sequence of finite graphs embedded in a domain  $D \subseteq \mathbb{C}$ . Let  $z^{\#\delta}$  be a vertex in  $G^{\#\delta}$  closest to a point  $z \in D$ . Let  $t_0 \in [-\infty, \infty)$  be large enough so that  $e^{-t_0} < R(z, D)/10$ . We assume that for  $\delta < \delta_B$  where  $\delta_B$  depends only on  $B := B(z, e^{-t_0})$ , the crossing condition holds for all rectangles in  $B$  which are translations of  $\delta R$  or  $\delta R'$  where  $R, R'$  are as in assumption v. We take  $t_0 = -\infty$  if  $D = \mathbb{C}$ .

Fix  $t_0 < m \in \mathbb{Z}$  and  $t > 10t_0$ . Let  $C_i$  denote  $B(z, e^{-i})$  for  $m \leq i \leq t+2$ . Let  $\{\tau_k\}$  be inductively defined as in the proof of Proposition 4.10. We have  $\tau_0 = 0$  and  $\tau_1$  is the time the random walk  $X$  crosses or hits  $C_{t+2}$ . Having defined  $\tau_k$  to be the smallest time when the random walk  $X$  crosses (or hits) some  $C_j$ , we define  $\tau_{k+1}$  to be the smallest time when it hits  $C_{j-1}$  or  $C_{j+1}$ . If  $j = m$  or  $t+2$ , we define  $\tau_{k+1}$  to be the smallest time the walk crosses or hits  $C_{m+1}$  or  $C_{t+1}$ . Let  $Y^k$  denote the loop erasure of  $X[0, \tau_k]$  with some arbitrary continuous monotone parametrisation starting from  $\tilde{v}$ . Let  $\{\tau_{i,j}\}_{j \geq 1}$  be the sequence of times when the walk crosses or hits  $C_m$ . Let  $t_{1,j}$  be the last time when  $Y^{i,j}$  exits  $B(\tilde{v}, e^{-t-1})$  and let  $t_{2,j}$  be the last time when  $Y^{i,j}$  exits  $B(\tilde{v}, e^{-t})$ . Let  $[u_1 = t_{1,j}, v_1], [u_2, v_2], [u_3, v_3], \dots, [u_p, v_p = t_{2,j}]$  be the sequence of maximal disjoint intervals when  $Y^{i,j}[t_{1,j}, t_{2,j}]$  is inside  $B(z, e^{-t_0})$ .

**Proposition 4.14.** *Consider the above setup. There exists  $\alpha \in (0, 1)$  and  $C, c' > 0$  such that the following holds. For all  $t$  as above and  $j \geq 1$  there exists  $\delta_0 < \delta_B$  such that for all  $n \geq 0$ ,  $\delta < \delta_0$ ,*

$$\mathbb{P}\left(\sum_{\ell=1}^p \sup_{\mathcal{Y} \subseteq Y^{i,j}[u_\ell, v_\ell]} |W(\mathcal{Y}, \tilde{v})| > n\right) < C e^{-c' n^\alpha}.$$

*In particular, all moments of  $\sum_{\ell=1}^p \sup_{\mathcal{Y} \subseteq Y^{i,j}[u_\ell, v_\ell]} |W(\mathcal{Y}, \tilde{v})|$  exist and are finite.*

*Proof.* The proof is identical to the proof of Proposition 4.10 except instead of waiting for the random walk to exit a domain we have to wait for the walk to hit or cross the circle  $C_m$  for  $j$  times. Since the size of the domain did not play a role in the proof of Proposition 4.10, the integer  $j$  plays no role in the bound as claimed in the proposition. Thus identical arguments yield that the quantities  $I$  and  $\mathcal{G}_j$  have stretched exponential tail (if  $D \neq \mathbb{C}$ , we only have finitely many annuli to consider outside  $B$ ). Since the exponential tail on the winding of random walk in the annuli is still valid under the crossing assumption inside  $B$ , we obtain that the winding of each elementary piece of random walk as defined in the proof of Proposition 4.10 has exponential tail. Since the portion of the loop erased walk inside  $B$  is a subset of the random walk inside  $B$ , we conclude using Lemma 4.9.  $\square$

#### 4.4 Local coupling of spanning trees

Let  $z_1, \dots, z_k \in D^{\#\delta}$ . The goal of this section is to establish a coupling between a wired uniform spanning tree  $\mathcal{T}^{\#\delta}$  in  $D^{\#\delta}$  and  $k$  independent copies of full plane spanning tree measure  $\mathcal{T}_i$  such that for all  $i$ , there is a neighbourhood  $N_i$  around  $z_i$  on which  $\mathcal{T}^{\#\delta}$  match with  $\mathcal{T}_i^{\#\delta}$ . The diameter



of the neighbourhoods  $N_i$  are going to be random but nevertheless we will have a good bound on the probability of the diameter being very small.

The overall strategy will be to sample the spanning tree  $\mathcal{T}^{\#\delta}$  in  $D^{\#\delta}$  and the  $\mathcal{T}_i^{\#\delta}$  using Wilson algorithm. The coupling will mostly be achieved by using the same random walk for  $\mathcal{T}^{\#\delta}$  and the  $\mathcal{T}_i^{\#\delta}$ . To achieve independence and obtain the tail estimate for the diameters of the neighbourhoods  $N_i$ , we will choose carefully the points from which we sample loop-erased walks and keep track of the distances from  $\{z_i\}$  to the sub-tree discovered at any step.

We start with a simple lemma regarding hitting probability of random walk. This is rewriting Lemma 2.1 of Schramm [39] in our setting.

**Lemma 4.15.** *Let  $K \subset \mathbb{C}$  be a connected set such that the diameter (in the metric inherited from the Euclidean plane) of  $K$  is at least  $R$ . Then there exists a universal constant  $C > 0$  such that for all  $\delta \in (0, C \operatorname{dist}(v, K)\delta_0)$ , there exist universal constants  $c_0$  and  $c_1$  such that*

$$\mathbb{P}(\text{simple random walk from } v \text{ exits } B(v, R)^{\#\delta} \text{ before hitting } K^{\#\delta}) \leq c_0 \left( \frac{\operatorname{dist}(v, K)}{R} \right)^{c_1}$$

*Proof.* Let  $C_i$  denote the circle of radius  $2^{-i}$  around  $v$  for  $i \in \mathbb{Z}$ . Consider a sequence of stopping times  $\{T_k\}_{k \geq 0}$  defined as in Proposition 4.8: if  $T_k$  is the time when the walk crosses  $C_i$  then  $T_{k+1}$  is the smallest time after  $T_k$  when the simple random walk crosses  $C_{i+1}$  or  $C_{i-1}$ . The number of circles which intersect  $K$  is at least  $c \log_2(\frac{R}{\operatorname{dist}(v, K)})$  for some  $c > 0$ . The choice of  $\delta$  is small enough for Lemma 4.3 to apply for the annuli bounded by these circles. Whenever the walk at  $T_k$  is in a circle  $C_i$  such that both  $C_{i-1}$  and  $C_{i+1}$  are subsets of  $D$  and intersect  $K$ , then the walk has probability at least  $\alpha > 0$  of performing a full turn in  $A(v, 2^{-i-1}, 2^{-i+1})$  via Lemma 4.3. But doing such a full turn implies the walk must hit  $K$ . Hence the probability of the walk exiting  $D^{\#\delta}$  without hitting  $K$  has probability at most  $(1 - \alpha)^{c' \log_2(\frac{R}{\operatorname{dist}(v, K)})}$  for some  $c' > 0$  which concludes the proof.  $\square$

Let  $D$  be a fixed domain, let  $v \in D^{\#\delta}$ , and let  $r$  be such that  $B(v, r) \subset D$ . Using Wilson's algorithm, we now prescribe a way to sample the portion of the wired uniform spanning tree  $\mathcal{T}^{\#\delta}$  of  $D^{\#\delta}$  which contains all the branches emanating from vertices in  $B(v, r/2)^{\#\delta}$ . Consider  $\{\frac{r}{2}6^{-j}\mathbb{Z}^2\}_{j \geq 0}$ , a sequence of scalings of the square lattice  $\mathbb{Z}^2$  which divides the plane into square cells. At step  $j$ , pick a vertex from each cell which is farthest from  $v$  in  $B(v, \frac{r}{2}(1+2^{-j}))^{\#\delta} \cap \frac{r}{2}6^{-j}\mathbb{Z}^2$  (break ties arbitrarily) and is not chosen in any previous step. Call  $\mathcal{Q}_j$  the set of vertices picked in step  $j$ . Now sample branches of  $\mathcal{T}^{\#\delta}$  from each of these vertices in any prescribed order via Wilson's algorithm, resulting in a partial tree  $\mathcal{T}_j^{\#\delta}$ . We continue until we exhaust all the vertices in  $B(v, r/2)^{\#\delta}$ . We call this algorithm the **good algorithm**  $GA_D^{\#\delta}(r, v)$  to sample the portion of  $\mathcal{T}^{\#\delta}$  containing all branches emanating from vertices in  $B(v, r/2)^{\#\delta}$  (and in particular, the restriction of  $\mathcal{T}^{\#\delta}$  to  $B(v, r/2)^{\#\delta}$ ). Note in particular that  $GA_D^{\#\delta}(r, v)$  is sure to terminate after step  $j = \log_6(Cr/\delta)$ , where  $C$  depends only on the constant appearing in the bounded density assumption (assumption i).

The next lemma is similar to Schramm's finiteness theorem from [39]. Roughly, this says that for all  $\varepsilon > 0$ , if we fix a  $\rho$  sufficiently small depending only on  $\varepsilon$ , and reveal the branches of the spanning tree at a finite number of points with density approximately  $\rho$ , then with high probability none of the remaining branches would have diameter greater than  $\varepsilon$ . Schramm's finiteness theorem



is originally stated for the diameter of the remaining branches of the spanning tree (which are loop-erased paths) but in fact the result holds for the underlying random walks themselves. Also the original theorem is interested in sampling the whole tree while we only want to sample  $\mathcal{T}^{\#\delta} \cap B(v, r)$  for some  $r$ . Since we will need these properties later on, we write it for completeness, but the proof is exactly the same as in [39].

**Lemma 4.16** (Schramm's finiteness theorem). *Fix  $\varepsilon > 0$  and let  $D, v, r$  be as above. Then there exists a  $j_0 = j_0(\varepsilon)$  depending solely on  $\varepsilon$  such that for all  $j \geq j_0$  and all  $\delta \leq 6^{-j_0} \delta_0 r$ , where  $\delta_0$  is as in item **v**, the following holds with probability at least  $1 - \varepsilon$ :*

- The random walks emanating from all vertices in  $\mathcal{Q}_j$  for  $j > j_0$  stay in  $B(v, r)$ .
- All the branches of  $\mathcal{T}^{\#\delta}$  sampled from vertices in  $\mathcal{Q}_j \cap B(v, r/2)$  for  $j > j_0$  until they hit  $\mathcal{T}_{j_0}^{\#\delta} \cup \partial D^{\#\delta}$  have Euclidean diameter at most  $\varepsilon r$ . More precisely, the connected components of  $\mathcal{T}^{\#\delta} \setminus \mathcal{T}_{j_0}^{\#\delta}$  within  $B(v, r/2)$  have Euclidean diameter at most  $\varepsilon r$ .

*Proof.* For  $j \geq 1$ , the number of vertices in  $\mathcal{Q}_j$  is at most  $c6^j$  where  $c$  is a universal constant. Let  $j_{\max} := \lfloor \log_6(\frac{C\delta_0 r}{\delta}) \rfloor$ . The choice of  $j_{\max}$  is such that for  $j \leq j_{\max}$  our uniform crossing assumption holds and in particular we can apply Lemma 4.15. Notice each vertex in  $D^{\#\delta}$  is within Euclidean distance  $4 \cdot 6^{-j} r$  from a vertex in  $\mathcal{Q}_{j-1}$ . By Lemma 4.15 and the choice of  $\delta$ , for  $j \leq j_{\max}$ , there exists a  $C_0$  such that the probability that the simple random walk from a vertex in  $\mathcal{Q}_j$  reaches Euclidean distance  $C_0 6^{-j} r$  from its starting point without hitting  $\mathcal{T}_{j-1}^{\#\delta}$  is at most  $1/2$ . Notice that  $j^2 6^{-j} < 2^{-j}$  for all  $j \in \mathbb{N}$  and hence using the Markov property, we can iteratively apply the same bound for the walk  $j^2/C_0$  times (this is the reason why in the good algorithm we sample from balls of decreasing size at each step). This shows that the probability that the random walk emanating from a vertex  $w$  in  $\mathcal{Q}_j$  has diameter greater than  $j^2 6^{-j} r$  (call this event  $\mathcal{D}(w, j)$ ) is at most  $(1/2)^{j^2/C_0}$ .

Recall that the bounds above hold for  $j \leq j_{\max}$ . When  $j > j_{\max}$  and  $w \in \mathcal{Q}_j$  we define  $\mathcal{D}(w, j)$  to be the event that the random walk emanating from  $w$  reaches distance  $j_{\max}^2 6^{-j_{\max}} r$  without hitting  $\mathcal{T}_j^{\#\delta}$  (and in particular  $\mathcal{T}_{j_{\max}}^{\#\delta}$ ). Then observe that we still have  $\mathbb{P}(\mathcal{D}(w, j)) \leq (1/2)^{j_{\max}^2/C_0}$  in this case. Notice that the number of lattice points in  $\cup_{j > j_{\max}} \mathcal{Q}_j$  is at most  $B6^{j_{\max}}$  for some constant  $B$  depending only on  $\delta_0$  in the uniform crossing estimate assumption and bounded density assumptions (items **i** and **v**).

Notice that on the complement of

$$\mathcal{D} := \cup_{j_0 \leq j} \cup_{w \in \mathcal{Q}_j} \mathcal{D}(w, j)$$

no random walk emanating from a vertex  $w \in \mathcal{Q}_j$  can reach distance  $j^2 6^{-j} r$  from its starting point and hence stays in  $B(v, r)$ . Furthermore, by a union bound,

$$\mathbb{P}(\mathcal{D}) \leq \sum_{j \geq j_0} c6^j (1/2)^{j^2/C_0} < \varepsilon \quad (4.6)$$

for large enough choice of  $j_0 = j_0(\varepsilon)$  which shows the first property of the lemma.

Also, let  $\mathcal{E}(w, j)$  be the event that a connection from  $w$  to  $\mathcal{T}_{j-1}^{\#\delta}$  has diameter at least  $j^2 6^{-j} r$ . Observe that conditionally on  $\mathcal{T}_{j-1}^{\#\delta}$ , the probability of the event  $\mathcal{E}(w, j)$  does not depend on the orders of the points in  $\mathcal{Q}_j$  and hence we may assume that  $w$  is the first point in  $\mathcal{Q}_j$  when we compute

this probability. In that case, the probability in question is at most the one we computed above, and we deduce that  $\mathbb{P}(\mathcal{E}(w, j) | \mathcal{T}_{j-1}^{\#\delta}) \leq (1/2)^{(j \wedge j_{\max})^2 / C_0}$ . Furthermore, on the complement of

$$\mathcal{E} := \bigcup_{j_0 \leq j} \bigcup_{w \in \mathcal{Q}_j} \mathcal{E}(w, j) \quad (4.7)$$

each point  $w \in \mathcal{Q}_j$  is connected to a point in  $\mathcal{T}_{j_0}^{\#\delta}$  by a path of diameter at most  $\sum_{j > j_0} j^2 6^{-j} r \leq \varepsilon r$  provided that  $j_0$  is large enough. Furthermore, as above  $\mathbb{P}(\mathcal{E}) \leq \varepsilon$  if  $j_0$  is large enough, which concludes the proof.  $\square$

Now we shall describe the coupling between a wired spanning tree in  $D^{\#\delta}$  and in a full plane. Recall that a priori it is not even clear that the full plane local weak limit of a wired spanning tree exists. For undirected graphs, the existence of this limit follows from the theory of electrical networks. However our setting includes directed graphs where the electrical network theory no longer applies. The existence of this limit will actually come out of our coupling procedure.

The full coupling is an iteration of the following coupling which we call the **base coupling**. Basically, the idea is the following. We wish to couple a uniform spanning tree in  $D^{\#\delta}$  to a uniform spanning tree in  $\tilde{D}^{\#\delta}$  within a neighbourhood of some fixed vertex  $v$ . To do this, we first make sure that the branches from a finite number of vertices coincide for UST's in  $D$  and  $D'$  in some neighbourhood, and then we apply Schramm's finiteness theorem. We now explain this in more details.

**Base coupling.** The base coupling we describe now takes the following parameters as input: Two domains  $D, \tilde{D}$  and a neighbourhood  $B(v, 10r)$  of a vertex  $v$  such that  $B(v, 10r) \subset D \cap \tilde{D}$ . Let  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  denote a sample of uniform spanning tree in  $D^{\#\delta}$  and  $\tilde{D}^{\#\delta}$  respectively. For any vertex  $u$  in  $D^{\#\delta}$  (resp.  $\tilde{D}^{\#\delta}$ ), let  $\gamma(u)$  (resp.  $\tilde{\gamma}(u)$ ) denote the branch of  $\mathcal{T}$  (resp.  $\tilde{\mathcal{T}}$ ) from  $u$  to the boundary of  $D^{\#\delta}$  (resp.  $\tilde{D}^{\#\delta}$ ).

(i) Pick a point  $u_1$  in  $A^{\#\delta}(v, 8r, 9r)$  and sample  $\gamma(u_1), \tilde{\gamma}(u_1)$  independently. Let  $\mathcal{E}_1$  be the event that both  $\gamma(u_1)$  and  $\tilde{\gamma}(u_1)$  stay outside  $B(v, 7r)$  and suppose  $\mathcal{E}_1$  holds.

(ii) Let  $u_2 \in A^{\#\delta}(v, 2r, 3r)$ . We will use the same underlying random walk to couple  $\gamma(u_2)$  and  $\tilde{\gamma}(u_2)$ . More precisely, start a simple random walk from  $u_2$  until it is in one of  $\gamma(u_1) \cup \partial D^{\#\delta}$  or  $\tilde{\gamma}(u_1) \cup \partial \tilde{D}^{\#\delta}$  at time  $t_1$ . Suppose without loss of generality that the walk hits  $\gamma(u_1) \cup \partial D^{\#\delta}$  at  $t_1$ . Then we continue the walk from that point until it hits the other path or the boundary at time  $t_2$ . We then define  $\gamma(u_2)$  to be the loop erased path up to time  $t_1$  and  $\tilde{\gamma}(u_2)$  to be the loop erased path from time 0 to  $t_2$ .

Let  $\mathcal{E}_2$  be the event that  $\gamma(u_2)$  and  $\tilde{\gamma}(u_2)$  agree in  $B(v, 6r)$ , and suppose  $\mathcal{E}_2$  holds.

(iii) Fix a  $j_0 = j_0(1/2)$  as defined in Lemma 4.16. Let  $\mathcal{Q}_j$  be a set of points, one in each cell of  $6^{-j}(r/2)(1 + 2^{-j})\mathbb{Z}^2 \cap D$  which is farthest away from  $v$ , as described in the good algorithm above. Let  $\mathcal{E}_3$  be the event that the branches from all the vertices in  $\bigcup_{j \leq j_0} \mathcal{Q}_j$  of  $\mathcal{T}, \tilde{\mathcal{T}}$  agree in  $B(v, 5r)$ , and suppose that  $\mathcal{E}_3$  holds.

(iv) Finally let  $\mathcal{E}_4$  be the event that all the branches from vertices in  $\bigcup_i \mathcal{Q}_i$  of  $\mathcal{T}, \tilde{\mathcal{T}}$  agree in  $B(v, r)$ .

In the steps above if  $\bigcap_i \mathcal{E}_i$  does not occur, we say that the **base coupling has failed**. We think of the above process as sampling branches one by one from the prescribed vertices. We stop this process of sampling branches at the first time a sampled branch causes the base coupling to fail.

**Lemma 4.17.** *There exist universal constants  $0 < p_0 < p'_0 < 1$  and  $C > 0$  such that for all  $\delta \leq C\delta_0 r$ ,*

$$p_0 < \mathbb{P}(\text{base coupling succeeds in } D^{\#\delta}) < p'_0$$

*Proof.* The proof essentially follows from Proposition 4.8 which says that loop erased random walk does not come too close to a particular vertex and Lemma 4.3 which says that random walk makes a full turn in a given annulus with positive probability. To start with, it follows from Proposition 4.8 (possibly replacing 4 there by some other number) that  $\mathcal{E}_1$  has a positive probability  $p_1$ . Also using the crossing estimate it is easy to see that  $\mathbb{P}(\mathcal{E}_1) < p'_0 < 1$  which completes the proof of the upper bound. Moreover, independently of  $\mathcal{E}_1$ , the walk started from  $u_2$  after exiting  $B(v, 7r)$  has probability at least  $p_2$  to make a full turn in  $A(v, 9r, 10r)$  without first hitting  $B(v, 6r)$ , by Lemma 4.3. In particular, this implies  $\mathcal{E}_2$  has probability at least  $p_2$ , conditionally on  $\mathcal{E}_1$ .

Now assume  $\mathcal{E}_1 \cap \mathcal{E}_2$  holds. Let  $w \in \mathcal{Q}_j$  for  $j \leq j_0$ , and assume that revealing previous branches did not make the coupling fail. Then the walk started from  $w$  has a positive probability  $p_3$  to do a full turn in  $A(v, 3r, 5r)$  before leaving  $B(v, 5r)$ . If this occurs then the corresponding branches  $\gamma(w)$  and  $\tilde{\gamma}(w)$  will agree in  $B(v, 5r)$  (since the walk is then certain to hit at least both  $\gamma(u_2)$  and  $\tilde{\gamma}(u_2)$  in that ball). Iterating this bound over a bounded number of points (of order  $6^{j_0}$ ) shows that, conditionally on  $\mathcal{E}_1 \cap \mathcal{E}_2$ ,  $\mathcal{E}_3$  has uniformly bounded below probability.

Finally, conditionally on  $\cap_{1 \leq i \leq 3} \mathcal{E}_i$ ,  $\mathcal{E}_4$  has probability at least  $1/2$  by Schramm's finiteness theorem (Lemma 4.16), which finishes the proof.  $\square$

The general idea for the full coupling around one point  $v$  will be that when the base coupling fails there is a not too small neighbourhood around  $v$  which was not intersected by any of the paths we sampled so far. Therefore we will be able to retry the coupling in a new smaller neighbourhood. For the coupling around several points, the idea will be similar. We will show that after doing the full coupling around one point, there are still “unexplored” neighbourhoods around the other points on which we can apply our one-point results.

To implement this strategy we first show that if the base coupling fails,  $v$  remains reasonably isolated at the point when we stop the process with high probability. We say a vertex  $u$  has **isolation radius**  $6^{-k}$  at scale  $r$  at any step in the above base coupling (centered around  $v$ ) if

$$B(u, 6^{-k}r) \text{ does not contain any vertex from a sampled branch.}$$

We then set  $I_u$  to be the minimum such  $k \geq 1$  at the time when the base coupling fails.

**Lemma 4.18.** *Let  $I_u$  be as above and suppose that either  $u = v$  or  $|u - v| \geq 10r$ . Then there exists a universal constant  $\delta' > 0$  such that for all  $\delta \in (0, \delta' r)$  and for all  $i \in (0, \log_6(\delta' r / \delta) - 1)$ , there exist positive universal constants  $c, c' > 0$  such that*

$$\mathbb{P}(I_u \geq i \mid \text{coupling fails}) \leq ce^{-c'i}$$

*Proof.* From Proposition 4.8, if one of  $\mathcal{E}_k$  fails for  $k = 1, 2, 3$ , the probability that the isolation radius is at least  $i$  is at most  $C\alpha^i$  for some  $C > 0$  and  $\alpha \in (0, 1)$  (since it is the maximum of a finite number of variables each with exponential tail). The final case we need to consider is the branches we draw while doing the good algorithm in item iv of the base coupling.

Notice that the number of vertices in  $\mathcal{Q}_j$  for  $j \geq j_0$  is at most  $C_0 6^j$  and each of them is at a distance at least  $r6^{-j-1}$  from  $u$  (note that this holds both when  $u = v$  or  $|u - v| \geq 10r$ ). Let  $\mathcal{A}(i, j)$  be the event that coupling fails in step  $j \geq j_0$  and  $I_u$  after this step is at least  $i$ . For

$i \in (j^2, \log_6(\delta_0 r / \delta))$ , probability of  $\mathcal{A}(i, j)$  is at most the probability that the branch  $\gamma(w)$  from one of the vertices  $w \in \mathcal{Q}_j$  comes within distance  $6^{-i}r$  of  $u$ . By Proposition 4.8 and a union bound, this is bounded by  $C_0 6^j \alpha^{i-j} \leq C_0 6^{\sqrt{i}} \alpha^{i-\sqrt{i}}$ , which decays exponentially even when we sum over  $j \leq \sqrt{i}$ .

Finally if  $i < j^2$ , we bound the probability of  $\mathcal{A}(i, j)$  by using the explicit error bound eq. (4.6) which we obtained in the proof of Schramm's finiteness theorem: indeed, we showed that the probability one of the branches emanating from a vertex  $w \in \mathcal{Q}_j$  leaves the ball of radius  $j^2 6^{-j}$  around  $w$  is less than  $C 6^j \alpha^{j^2}$ . In particular this is also a bound on the probability that one of these branches leaves  $B(v, r)$ . Hence in that  $\mathbb{P}(\mathcal{A}(i, j)) \leq C 6^j \alpha^{j^2} \leq C \alpha'^{j^2}$  for some  $\alpha' < 1$ . Summing over  $j \geq \sqrt{i}$  we get  $\mathbb{P}(I_u \geq i) \leq C \alpha'^i$ .

It remains to condition on the event that the coupling fails. But since the coupling fails with probability bounded below by Lemma 4.17, the result follows.  $\square$

**Iteration of base coupling:** We now describe how to iterate the base coupling at different scales to construct the full coupling. We start with a domain  $D \subset \mathbb{C}$  and suppose  $v_1, v_2, \dots, v_k \in D^{\#\delta}$ . Suppose  $r < 1$  is small enough such that  $B(v_i, 10r)$  are disjoint and contained in  $D \cap \tilde{D}$ . Fix a small constant  $C$  (which will be fixed later) and assume that  $\delta \leq C \delta_0 r$ .

- (i) Perform a base coupling with  $D^{\#\delta}, \tilde{D}^{\#\delta}$  and  $B(v_1, r)$ . If the coupling succeeds, we move on to vertex  $v_2$ .
- (ii) If the coupling fails, let  $6^{-I_{v_1,1}}r$  be the isolation radius of  $v_i$  at scale  $r$  at the step the coupling has failed. If  $\max_{1 \leq i \leq k} I_{v_i,1} \geq \log_6(C \delta_0 r / \delta)$ , we **abort** the whole process and we say that the full coupling failed.
- (iii) If the coupling has failed but we haven't aborted, we move to a smaller scale. Let  $\mathcal{T}_{1,1}$  be the portion of the uniform spanning tree in  $D$  sampled up to this point. We perform the base coupling in the domains  $D^{\#\delta} \setminus \mathcal{T}_{1,1}, \tilde{D}^{\#\delta}$  in the ball  $B(v_1, 6^{-I_{v_1,1}}r)$  around  $v_1$ . (We emphasise here that we perform this step with a fresh independent sample of uniform spanning tree of  $\tilde{D}^{\#\delta}$ . Hereafter if the coupling fails at any step, we sample a new independent copy of the spanning tree in  $\tilde{D}^{\#\delta}$ ).
- (iv) If the coupling fails, let  $6^{-I_{v_1,1}-I_{v_1,2}}r$  be the isolation radius at scale  $r$  around  $v_i$  at the step the coupling has failed. Let  $\mathcal{T}_{1,2} \supset \mathcal{T}_{1,1}$  be the uniform spanning tree of  $D^{\#\delta}$  sampled up to this point. If  $\max_{1 \leq i \leq k} (I_{v_i,1} + I_{v_i,2}) \geq \log_6(C \delta_0 r / \delta)$ , we abort the whole process. Otherwise we perform the base coupling with  $D^{\#\delta} \setminus \mathcal{T}_{1,2}, \tilde{D}^{\#\delta}$  and  $B(v_1, 6^{-(I_{v_1,1}+I_{v_1,2})}r)$ .
- (v) We continue in this way until we either abort or the base coupling succeeds at the  $N_1$ th iteration. If we haven't aborted the process along the way, we have obtained a partial tree  $\mathcal{T}_{1,N_1}$  which is coupled with an independent sample of the uniform spanning tree in  $D_1^{\#\delta}$  in  $B(v_1, 6^{-(I_{v_1,1}+I_{v_1,2}+\dots+I_{v_1,N_1-1})}r)$ .
- (vi) After succeeding around  $v_1$ , we move to  $v_2$  restarting the whole procedure in  $D^{\#\delta} \setminus \mathcal{T}_{1,N_1}$  and  $\tilde{D}^{\#\delta}$  in the neighbourhood  $B(v_2, 6^{-(I_{v_2,1}+\dots+I_{v_2,N_1})}r)$  of  $v_2$ .
- (vii) For a generic step: let  $I = \sum_{\ell=1}^{N_1} I_{v_1,\ell} + \dots + \sum_{\ell=1}^{N_{i-1}} I_{v_{i-1},\ell} + I_{v_i,1} + \dots + I_{v_i,j}$  so that  $6^{-I}r$  is the isolation radius around  $v_i$  after the  $j$ th iteration of the above process. When we are doing the base coupling around  $v_i$ , suppose the portion of the tree  $\mathcal{T}$  sampled after step  $j$  is  $\mathcal{T}_{i,j}$ .

At the  $(j+1)$ th step we perform the base coupling in  $D^{\#\delta} \setminus \mathcal{T}_{i,j}$ ,  $\tilde{D}^{\#\delta}$  and  $B(v_i, 6^{-I_i}r)$ . If the coupling succeeds, we declare  $N_i = j+1$ , we call  $\mathcal{T}_{i,N_i} = \mathcal{T}_{i+1,0}$  the resulting tree, we move on to  $v_{i+1}$  (if  $i = k$  we finish). If  $I + I_{v_i,j+1} \geq \log_6(C\delta_0 r/\delta)$ , we abort. If the base coupling fails and we haven't aborted, we move on to the next scale and iterate (vii) to build  $\mathcal{T}_{i,j+1}$ .

- (viii) If we abort the process at step  $m$  while performing base coupling around vertex  $v_i$ , define  $N_i = N_{i+1} = \dots = N_k = m$  by convention.

We call  $I_i = \sum_{\ell=1}^{N_1} I_{v_1,\ell} + \dots + \sum_{\ell=1}^{N_i-1} I_{v_i,\ell}$ , so that  $6^{-I_i}r$  is the isolation radius at scale  $r$  when we have succeeded in coupling the trees around  $v_i$ .

**Theorem 4.19.** *On the event  $\mathcal{A}$  that we do not abort the full coupling, we obtain a coupling between  $\mathcal{T}^{\#\delta}$  and independent copies of uniform spanning trees  $\tilde{\mathcal{T}}^{\#\delta}(j)$  in  $\tilde{D}^{\#\delta}$  for  $1 \leq j \leq k$  such that*

$$\mathcal{T}^{\#\delta} \cap B(v_i, 6^{-I_i}r) = \tilde{\mathcal{T}}^{\#\delta}(j) \cap B(v_i, 6^{-I_i}r)$$

Furthermore, there exists a universal constant  $C > 0$  and such that for all  $\delta \leq C\delta_0 r$  and  $1 \leq i \leq k$ ,

$$\mathbb{P}(I_i \geq n; \mathcal{A}) \leq ce^{-c'n} \quad (4.8)$$

for some universal constants  $c, c' > 0$ .

*Proof.* Observe that  $I_i$  is a sum of  $i-1$  terms of the form  $\sum_{\ell=1}^{N_j} I_{v_j,\ell}$  and a final term of the form  $\sum_{\ell=1}^{N_i-1} I_{v_i,\ell}$ . We claim each of these  $i$  terms has a geometric tail: indeed, observe that by Lemma 4.17,  $N_j$  has geometric tail (since the coupling has probability bounded below to succeed at every step, independently of the past). Moreover, each  $I_{v_j,\ell}$  has uniform exponential tail conditionally on all previous steps by Lemma 4.18. The proposition follows easily.  $\square$

The following two consequences are immediate:

**Corollary 4.20.** *The probability of aborting the above process is at most  $c(\delta/r\delta_0)^{c'}$  for some universal constants  $c, c' > 0$ .*

*Proof.* The event  $\mathcal{A}^c$  occurs precisely when one of the variables  $I$  in step (vii) exceeds  $\log_6(Cr\delta_0/\delta)$ . For the same reason as above, this has exponential tail.  $\square$

**Corollary 4.21.** *The wired uniform spanning tree measures in  $D^{\#\delta}$  has a local limit when  $\tilde{D} \rightarrow \mathbb{C}$  and this limit is independent of the exhaustion taken. We call this measure the **whole plane spanning tree**. In Theorem 4.19 and all the above statements, the spanning tree measure on  $\tilde{D}^{\#\delta}$  can be replaced by a whole plane spanning tree.*

*Proof.* For the first sentence, consider an exhaustion  $D_n$  i.e., an increasing sequence such that  $\cup_n D_n = \mathbb{C}$ . Using Theorem 4.19 along with the control of the abortion probability from Corollary 4.20 we conclude that the spanning tree measures form a Cauchy sequence in total variation, therefore it converges. For the second one, it then follows immediately from the fact that all the statements are uniform on domains  $\tilde{D}$ .  $\square$

**Remark 4.22.** Suppose that  $(\gamma, \tilde{\gamma})$  are coupled by a global coupling as above. Let  $T_i$  be the smallest time  $t$  such that  $\gamma(t) \in B(z_0, e^{-i})$ , and let  $T = T_I$ . Then given  $I = i$  the law of  $(\gamma_t, t \geq T)$  is absolutely continuous with respect to the unconditional law of  $(\gamma_t, t \geq T_i)$  with Radon-Nikodym

derivative bounded by  $C$  for some universal constant  $C > 0$  as we vary  $\delta$  (indeed, it is just the law of a loop-erased random walk conditioned on some event of uniformly positive probability).

Also, given  $I = i$ , the law of the entire path  $\tilde{\gamma}$  is absolutely continuous with respect to the unconditional law, with Radon-Nikodym derivative bounded by  $C < \infty$ .

## 4.5 Comparisons of capacity

In the previous sections, we have proved that each branch of the continuum UST has one portion where the SLE approximation holds and another portion where the coupling holds between a path  $\gamma$  in  $\mathbb{D}$  and a path  $\tilde{\gamma}$  in the whole plane. A difficulty which arises here. For the SLE approximated portion we want to use the capacity parametrisation in order to have a direct connection with estimates we proved purely in the continuum. On the other hand, in view of the coupling result in the last section, it will be convenient to cut  $\gamma$  and  $\tilde{\gamma}$  at exactly the same point. However the capacity parametrisation of  $\gamma$  and  $\tilde{\gamma}$  do not agree exactly. In this section we tackle this issue (see e.g. Lemma 4.26). The idea is that the capacities of  $\gamma$  and  $\tilde{\gamma}$  are not only close for large values of capacity, the rate of increase of capacities are also close. We can thus couple Poisson processes with very similar intensities in the two capacities so that the first jump matches up and we cut at the first jump time.

We first need some technical estimates on the comparison of the two capacities. In this section  $D$  will be a fixed domain.  $\gamma$  and  $\tilde{\gamma}$  will always denote two simple paths from a point  $z_0 \in D$  to respectively  $\partial D$  and infinity and we will always assume that there exists  $I$  such that  $\gamma \cap B(z_0, e^{-I}) = \tilde{\gamma} \cap B(0, e^{-I})$ . For points  $z, z' \in \gamma$  we will write  $\gamma(z, z')$  for the part of the path between  $z$  and  $z'$ ,  $\gamma(z, \partial D)$  for the path between  $z$  and the boundary and similarly for  $\tilde{\gamma}$ . We denote by  $T$  and  $\tilde{T}$  the capacity in  $D$  or  $\mathbb{C}$ , i.e  $T(z) = -\log R(z_0, D \setminus \gamma(z, \partial D))$  and  $\tilde{T}(z) = -\log R(z_0, \mathbb{C} \setminus \tilde{\gamma}(z, \infty))$ , where recall that  $R(z, D)$  denotes the conformal radius of  $z$  in  $D$ .

The following lemma is elementary and well known.

**Lemma 4.23.** *Let  $D$  be a simply connected domain and  $z_0$  in  $D$ . Let  $B_t$  be a Brownian motion started in  $z_0$  and  $\tau$  the exit time of  $D$ . We have*

$$\log R(z_0, D) = \mathbb{E}(\log |B_\tau - z_0|).$$

**Lemma 4.24.** *There exist positive constants  $C$  and  $C'$  such that for any pair of paths  $\gamma$  and  $\tilde{\gamma}$  and any  $z \in \gamma \cap B(z_0, e^{-I})$ , we have*

$$|T(z) - \tilde{T}(z)| \leq C(T(z) - I)e^{-C'(T(z) - I)}.$$

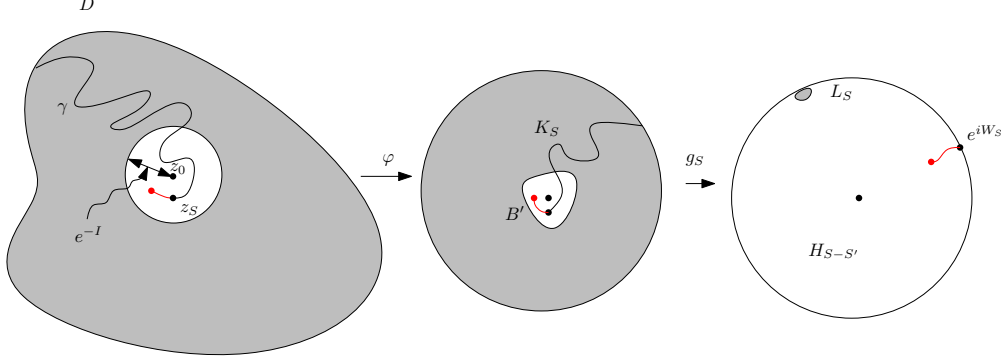
*Proof.* Let  $B_t$  be a Brownian motion and let  $\tau$  and  $\tilde{\tau}$  denote respectively the hitting time of  $\partial D \cup \gamma(z, \partial D)$  and  $\tilde{\gamma}(z, \infty)$ . Let  $\tau_I$  denote the exit time of  $B(z_0, e^{-I})$ .

By Lemma 4.23,  $T(z) - \tilde{T}(z) = \mathbb{E}[-\log |B_\tau - z_0| + \log |B_{\tilde{\tau}} - z_0|]$  but the right hand side is 0 whenever  $\tau \leq \tau_I$ . Furthermore,  $d(z_0, \gamma(z, \partial D)) \leq e^{-T(z)}$  by Schwarz's lemma. Hence let  $z'$  be such that  $z' \in \gamma(z, \partial D)$  and such that  $|z' - z_0| \leq e^{-T(z)}$ . Then by the Beurling estimate (see [30, Theorem 3.76]),  $\mathbb{P}(\tau_I \leq \tau) \leq C\sqrt{|z' - z_0|/e^{-I}} \leq Ce^{-(1/2)(T-I)}$ .

$$|\mathbb{E} \log |(B_\tau - z_0)/(B_{\tilde{\tau}} - z_0)| \mathbb{1}_{\tau \geq \tau_I}| \leq \sqrt{\mathbb{E} \log^2 |(B_\tau - z_0)/(B_{\tilde{\tau}} - z_0)| \mathbb{P}(\tau \geq \tau_I)} \leq C(T(z) - I)e^{-c'(T(z) - I)}$$

which completes the proof.  $\square$





**Figure 12:** Sketch of the maps in the proof of Lemma 4.25

Now we will compare the growth rate of the capacities of  $\gamma$  and  $\tilde{\gamma}$ . We will also need to introduce  $\bar{T}(z) = -\log R(z_0; B(z_0, e^{-I}) \setminus \gamma(z, \partial D))$ . In other words this is the capacity of  $\gamma(z, \partial D)$  within  $B(z_0, e^{-I})$ . The next lemma depends on the geometry of the curves so it is convenient to go to the unit disc. Let  $\varphi : D \rightarrow \mathbb{D}$  be the conformal map sending  $z_0$  to 0 and such that  $\varphi'(z_0) > 0$ .

We parametrise the curve  $\gamma$  by capacity: for  $T \geq T_0 = -\log R(z_0, D)$ , let  $z = z_T$  be the point on  $\gamma$  such that  $T(z) = T$  and define  $g_T : D \setminus \gamma(z_T, \partial D) \rightarrow \mathbb{D}$ . Let  $K_T = \varphi(\gamma(z_T, \partial D))$  for  $T \geq T_0$ . Then  $(K_T)_{T \geq T_0}$  is a growing family of hulls in the unit discs. Let  $(g_T)_{T \geq T_0}$  be the (radial) Loewner flow in  $\mathbb{D}$  and  $(W_T)_{T \geq T_0}$  be the driving function (with  $W_T \in \mathbb{R}$ ) corresponding to  $(K_T)_{T \geq T_0}$ . Also set  $B' = \varphi(B(z_0, e^{-I}))$ ,  $H_T = \mathbb{D} \setminus K_T$  and finally set  $L_T = g_T(H_T \setminus B')$ .

**Lemma 4.25.** *There exist positive constants  $A, a, \varepsilon_0$  such that the following holds. Let  $S$  and  $S'$  be fixed with  $0 < S' - S < \varepsilon_0$ . Let  $d = \inf_{S \leq T \leq S'} d(L_T, e^{iW_T})$ . Let  $w = z(S)$  and  $w' = z(S')$ . If  $S - I$  is large enough and if  $d$  is sufficiently large so that both  $d \geq Ae^{-\frac{a}{4}(S-I)}$  and  $d^2 \geq A(S' - S)$  hold, then*

$$1 - Ae^{-a(S-I)}/d^2 < \frac{\bar{T}(w') - \bar{T}(w)}{S' - S} < 1 + Ae^{-a(S-I)}/d^2.$$

*Proof.* Let  $\rho$  be the image of  $\gamma(w', w)$  under  $g_S \circ \varphi$ , i.e, the red curve in the rightmost image of Figure 12. First we note that by definition

$$\bar{T}(w) - S = -\log R(0; \mathbb{D} \setminus L_S), \quad (4.9)$$

$$\bar{T}(w') - S = -\log R(0; \mathbb{D} \setminus (L_S \cup \rho)), \quad (4.10)$$

$$S' - S = -\log R(0; \mathbb{D} \setminus \rho). \quad (4.11)$$

Let  $\tau_\rho$ ,  $\tau_L$  and  $\tau_\partial$  denote the hitting time of respectively  $\rho$ ,  $L_S$  and  $\partial\mathbb{D}$  by a Brownian motion  $B$  starting from zero. Combining (4.10), (4.9), (4.11) and Lemma 4.23, we have

$$\bar{T}(w') - \bar{T}(w) - (S' - S) = \mathbb{E}^0 [-\log|B_{\tau_\partial \wedge \tau_L \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_L}|]. \quad (4.12)$$

Hence it suffices to show that

$$|\mathbb{E}^0 [-\log|B_{\tau_\partial \wedge \tau_L \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_L}|]| \leq \frac{Ce^{-c(S-I)}}{d^2} (S' - S). \quad (4.13)$$



Notice that if  $\tau_\partial < \tau_L \wedge \tau_\rho$  then the random variable in the right hand side of (4.12) is zero. We will consider separately the two cases where  $\tau_L$  is smallest and also  $\tau_\rho$  is smallest below in Steps 2 and 3 respectively, but in step 1, we establish some geometric estimates on  $L_S$  and  $\rho$ .

**Step 1.** First we prove that the distance between  $L_S$  and  $\rho$  is at least  $d/10$  for small enough  $S' - S$ . From the choice of  $d$ , we can draw an arc  $I_\ell$  to the left of the leftmost point in  $L_S \cap \partial D$  and  $I_r$  to the right of the rightmost point in  $L_S \cap \partial D$  both of length at least  $d/4$  (and hence they do not intersect  $e^{iW_S}$ ). Let  $b \in L_S \cap \partial \mathbb{D}$  and let  $a', c$  be the two extremities of the arcs  $I_\ell, I_r$  which are farthest from  $L_S$ . Using the radial Loewner equation to  $h_t = g_{S+t} \circ g_S^{-1}$  at  $z = b$  for now we see that

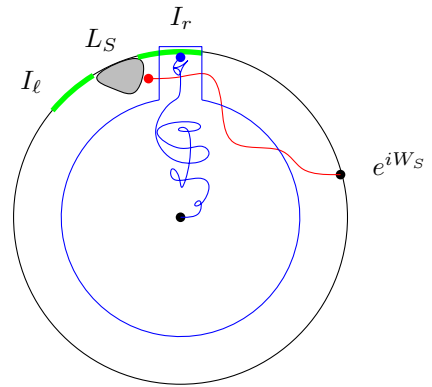
$$|\partial_t h_t| = |h_t \frac{e^{iW_{S+t}} + h_t}{e^{iW_{S+t}} - h_t}| \leq \frac{2}{d} \quad (4.14)$$

for all  $t \leq S' - S$  (this is because over this interval of time, we must have  $d(h_t(b), e^{iW_{t+S}}) \geq d$  and hence the denominator in the right hand side is greater than  $d$ ). Applying the same Loewner equation at  $z = a'$  and  $z = c$ , as long as  $d(h_t(a), e^{iW_{t+S}}) \geq d/2$ , and that  $t \leq S' - S$ , we get  $|\partial_t h_t| \leq 4/d$ . Hence  $|\partial_t h_t(a) - \partial_t h_t(b)| \leq 6/d$  on that interval. Combining with the fact that  $h_t(b)$  is at least  $d$  away from  $e^{iW_{t+S}}$  and eq. (4.14), we deduce that the time it would take for  $h_t(a')$  to be less than  $d/2$  away from  $e^{iW_{t+S}}$  is at least  $d^2/24$ . Hence if  $S' - S \leq d^2/C$  for some sufficiently large constant  $C > 24$ , then the condition  $d(h_t(a'), e^{iW_{t+S}}) \geq d/2$  is in fact always fulfilled throughout  $t \in [0, S' - S]$ . Observe that as  $S' - S \rightarrow 0$ ,  $d \rightarrow d(L_S, e^{iW_S}) > 0$  and hence we can always make such a choice. Consequently, we deduce that a bound similar to eq. (4.14) holds at  $z = a'$  and  $z = c$  with a different constant: for  $t \leq S' - S$ ,

$$|\partial_t h_t(z)| = |h_t(z) \frac{e^{iW_{S+t}} + h_t(z)}{e^{iW_{S+t}} - h_t(z)}| \leq \frac{4}{d}, \quad z = a', c. \quad (4.15)$$

Applying the above bound for the extremities of both the arcs  $I_\ell, I_r$  and integrating this bound over  $[0, S' - S]$  we see that  $h_{S'-S}(I_\ell)$  and  $h_{S'-S}(I_r)$  are arcs whose length is  $(d/4)(1 + O(1/C))$ . Hence this can be made arbitrarily close to  $d/4$ . Note that the lengths of  $h_{S'-S}(I_\ell)$  and  $h_{S'-S}(I_r)$  are nothing else but the harmonic measures in  $\mathbb{D} \setminus \rho$  of  $I_\ell, I_r$  respectively.

From this we can conclude that  $\rho$  does not come within distance  $d/10$  of  $L_S$ . Indeed, if it did, the harmonic measures of  $I_r$  or  $I_\ell$  would be small. More precisely, there is a universal constant  $c > 0$  such that conditionally on  $B_{\tau_\partial} \in I_r$  say,  $B$  crosses any set coming within distance  $d/10$  of  $L_S$  with probability at least  $c$ : for instance this necessarily happens if  $B$  stays in a certain deterministic set consisting of a union of a rectangle of dimensions  $d \times (d/20)$  and a disc of radius  $1 - d/2$  (see blue set in the accompanying Figure 13). This would imply that the harmonic measure of  $I_\ell$  or  $I_r$  in  $\mathbb{D} \setminus \rho$  is at most  $d/4 - c$  which is a contradiction to the fact that they can be made arbitrarily close to  $d/4$  by choosing  $S' - S$  small enough. Note also, for future reference, that  $h_{S'-S}(I_\ell)$  and  $h_{S'-S}(I_r)$  are arcs of length at least  $d/8$  and which don't intersect  $h(\rho)$  (since  $\rho$  does not hit  $\partial \mathbb{D}$ ).



**Figure 13:** Proof of step 1.

**Step 2.** Now we estimate the right hand side of (4.12), which was

$$\mathbb{E}^0 \left[ -\log|B_{\tau_\partial \wedge \tau_L \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_\rho}| + \log|B_{\tau_\partial \wedge \tau_L}| \right],$$

in the case where the Brownian motion hits  $\rho$  before  $L$  or  $\partial\mathbb{D}$ . In this case the first and second terms in the random variable of the right hand side cancel each other and we need to estimate  $\mathbb{E}^0(1_{\{\tau_\rho < \tau_\partial \wedge \tau_L\}} \mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial \wedge \tau_L}|))$ . Let  $\mathcal{C}$  denote the circle of radius  $d/10$  centered at  $B_{\tau_\rho}$ . Notice that by Step 1, since  $\rho$  does not come within distance  $d/10$  of  $L_S$ ,

$$|\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial \wedge \tau_L}|)| \leq |\log(1 - \text{Diam}(L_S))| \mathbb{P}^{B_{\tau_\rho}}(\tau_{\mathcal{C}} < \tau_\partial) \quad (4.16)$$

where  $\tau_{\mathcal{C}}$  is the stopping time when the Brownian motion hits  $\mathcal{C}$ . Now, observe that

$$\text{Diam}(L_S) \leq C \text{ Harm}_{\mathbb{D} \setminus L_S}(0; L_S) \leq C e^{-C'(S-I)} \quad (4.17)$$

by Beurling's estimate, where  $\text{Harm}_D(z, \cdot)$  denotes harmonic measure in  $D$  seen from  $z$ . Hence

$$\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial \wedge \tau_L}|) \leq C e^{-c'(S-I)} \mathbb{P}^{B_{\tau_\rho}}(\tau_{\mathcal{C}} < \tau_\partial)$$

using  $|\log(1-x)| = O(x)$ . Now it remains to bound  $\mathbb{P}^{B_{\tau_\rho}}(\tau_{\mathcal{C}} < \tau_\partial)$  from above. Let  $z = B_{\tau_\rho}$ . Then it is elementary to check that we have  $\mathbb{P}^z(\tau_{\mathcal{C}} < \tau_\partial) \leq c(1-|z|)/d$  for some constant  $c > 0$  (this is the probability of reaching distance  $d/10$  within a half-plane from  $i(1-|z|)$ : this can be seen by applying the map  $z \mapsto z^2$  and Beurling's estimate). Plugging this back into (4.16), we conclude:

$$|\mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial \wedge \tau_L}|)| \leq C e^{-c'(S-I)} \frac{1 - |B_{\tau_\rho}|}{d} \leq \frac{C e^{-c'(S-I)}}{d} |\log|B_{\tau_\rho}||.$$

Taking expectations with respect to  $\mathbb{E}^0$  on the event  $\tau_\rho < \tau_\partial \wedge \tau_L$ , we deduce that

$$|\mathbb{E}^0[1_{\tau_\rho < \tau_\partial \wedge \tau_L} \mathbb{E}^{B_{\tau_\rho}}(\log|B_{\tau_\partial \wedge \tau_L}|)]| \leq \frac{C e^{-c'(S-I)}}{d} \mathbb{E}^0|\log|B_{\tau_\rho}|| = \frac{C e^{-c'(S-I)}}{d} (S' - S),$$

which is slightly better than the bound  $(C e^{-c'(S-I)}/d^2)(S' - S)$  we are aiming for.

**Step 3.** Finally we consider the case where  $\tau_L < \tau_\partial \wedge \tau_\rho$ , which is the most delicate. If the Brownian motion hits  $L$  first then the first and third terms cancel each other and we need to estimate  $\mathbb{E}^0(\mathbb{E}^{B_{\tau_L}}(\log|B_{\tau_\partial \wedge \tau_\rho}|) 1_{\tau_L < \tau_\partial \wedge \tau_\rho})$  using the strong Markov property. We now claim that, almost surely,

$$\left| \frac{\mathbb{E}^{B_{\tau_L}}(\log|B_{\tau_\partial \wedge \tau_\rho}|)}{\mathbb{E}^0(\log|B_{\tau_\partial \wedge \tau_\rho}|)} \right| \leq \frac{C}{d^2} e^{-c'(S-I)} \quad (4.18)$$

To do this we are going use conformal invariance of harmonic measure and map out  $\rho$  via  $h_{S'-S}$ . Then we can use an explicit bound on the Poisson kernel on the disc which allows us to compare the harmonic measure of a set seen from 0 and from a point in  $L_S$ . By conformal invariance of harmonic measure, letting  $h = h_{S'-S}$

$$\begin{aligned} \mathbb{E}^{B_{\tau_L}}(-\log|B_{\tau_\partial \wedge \tau_\rho}|) &= - \int_{\rho} \log|z| \text{ Harm}_{\mathbb{D} \setminus \rho}(B_{\tau_L}, dz) \\ &= - \int_{h(\rho)} \log|h^{-1}(z)| \text{ Harm}_{\mathbb{D}}(h(B_{\tau_L}), dz) \\ &= - \int_{h(\rho)} \log|h^{-1}(z)| \frac{d \text{ Harm}_{\mathbb{D}}(h(B_{\tau_L}), dz)}{d \text{ Harm}_{\mathbb{D}}(0, dz)} \text{ Harm}_{\mathbb{D}}(0, dz) \end{aligned}$$

Observe that if we write  $h(B_{\tau_L}) = re^{i\theta} \in L_{S'}$ , then the Radon-Nikodym derivative above is simply the Poisson kernel and at the point  $z = e^{it}$  it is thus equal to  $P_r(\theta - t)$  where

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}$$

Note that when  $e^{it} \in h(\rho)$  then  $|\theta - t| \geq d/100$  by Step 1. Indeed, we know that  $h(I_\ell)$  and  $h(I_r)$  are arcs of length at least  $d/8$  which do not intersect  $h(\rho)$  (see the end remark of Step 1) and we know that the diameter of  $L_{S'}$  is at most  $Ce^{-C'(S-I)} \leq d/10^3$  by assumption for a choice of constants  $A > C, a < C'$  and large enough  $S - I$ . Hence the difference in arguments for a point in  $L_{S'}$  and a point in  $h(\rho)$  is at least  $d/100$ . Thus we have  $\cos(\theta - t) \leq 1 - d^2/10^5$ . Hence

$$P_r(\theta - t) \leq \frac{1 - r^2}{(1 - r)^2 + 2rd^2/10^5} \leq \frac{2Ce^{-C'(S-I)}}{d^2}$$

by our assumptions on  $d$ , the above choice of constants  $a, A$  and the fact that

$$1 - r \leq \text{Diam}(L_{S'}) \leq Ce^{-C'(S'-I)} \leq Ce^{-C'(S-I)}$$

by eq. (4.17). Since  $\mathbb{E}^0(\log|B_{\tau_\partial \wedge \tau_\rho}|) = S' - S$ , putting everything together we obtain Lemma 4.25.  $\square$

Taking the increment  $S' - S$  to 0 we get the following corollary:

**Lemma 4.26.** *Let  $a, A$  be as in Lemma 4.25. Let  $d = \inf_{S \leq T \leq S+1/10} d(L_T, e^{iW_T})$ . Let  $w$  and  $w'$  be the two points on  $\gamma$  corresponding respectively to  $S$  and  $S + 1/10$ . Let  $\mu$  and  $\bar{\mu}$  be the measures on  $\gamma$  obtained by capacity in  $D$  and  $B(z_0, e^{-I})$  respectively: that is, if  $z = z(T)$  and  $z' = z(T')$  then  $\mu(\gamma(z, z')) = |T' - T|$ , and likewise for  $\bar{\mu}$ . If  $S - I$  is large enough and if  $d \geq Ae^{-\frac{a}{4}(S-I)}$ , then, uniformly over  $\gamma(w, w')$ , we have*

$$1 - Ae^{-a(S-I)}/d^2 \leq \frac{d\bar{\mu}}{d\mu} \leq 1 + Ae^{-a(S-I)}/d^2.$$

Now we check that the assumption on  $d$  in the previous lemma holds when the curve is  $\text{SLE}_2$  with high probability.

**Lemma 4.27.** *Suppose  $\gamma$  is a radial  $\text{SLE}_2$  process independent of  $I$ . Define  $d$  for  $\gamma$  as in Lemma 4.26. For any  $a, A > 0$  we have*

$$d \geq Ae^{-a(S-I)} \text{ with probability } \geq 1 - c \exp(-c'(S - I)). \quad (4.19)$$

for some  $c, c' > 0$  depending only on  $A, a$ . Now suppose  $\gamma$  is a loop-erased random walk in  $D^{\# \delta}$  from  $z_0$  to  $\partial D$  and  $I$  as defined in Theorem 4.19 (we emphasise that  $I$  might not be independent of  $\gamma$ ). Let  $d$  be as above. Then there exists  $\delta = \delta(S)$  such that conditioned on  $I$ , if  $S - I$  is large enough and  $\delta \leq \delta(S)$ , for any  $a, A > 0$ , we have

$$d \geq Ae^{-a(S-I)} \text{ with probability } \geq 1 - \frac{c}{2} \exp(-c'(S - I)). \quad (4.20)$$

*Proof.* Let  $T$  be the smallest  $t$  such that  $\gamma(z(t)) \in B(z_0, e^{-I})$ , and let  $e^{i\theta_t}$  be one of the two images  $g_t \circ \varphi(z)$  for  $T \leq t \leq S$  where  $z = z(T)$ . Then note that  $e^{i\theta_S} \in \partial\mathbb{D} \cap L_S$ . Moreover we claim that with high probability, if  $S - I$  is large, then  $e^{i\theta_S}$  is far away from  $e^{iW_S}$ .

We can follow the evolution of  $e^{i\theta_t}$  under the Loewner flow  $g_t$  for  $T \leq t \leq S$  as follows. If  $Y_t = \theta_t - W_t$  then  $Y$  solves a stochastic differential equation:

$$dY_t = \cot(Y_t/2) - dW_t ; Y_T = 0.$$

(see (6.13) in [30]). By comparing with a Bessel process (of dimension 5 in the case  $\kappa = 2$ ), we can see that the probability for  $Y_t$  to hit  $[0, \delta]$  in any particular interval of length  $O(1)$  is polynomial in  $\delta$ . Thus for any given  $S$  such that  $S - I$  is large enough, then for any  $c, C$ ,

$$\mathbb{P}\left(\min_{S \leq t \leq S+1/10} Y_t \leq 10e^{-c(S-I)}\right) \leq C' \exp(-c'(S-I))$$

is exponentially small in  $S - I$ . Now, since  $\text{Diam}(L_S) \leq Ce^{-c(S-I)}$  by Beurling's estimate (see eq. (4.17)) we see that

$$d \geq Ce^{-C(S-I)} \text{ with probability } \geq 1 - C' \exp(-c'(S-I)).$$

This proves eq. (4.19).

To deduce eq. (4.20), we recall that loop-erased random walk converges to  $\text{SLE}_2$  under our assumptions (see Remark 4.1). Then use Remark 4.22 to note that conditionally on  $I = i$ , the law of  $\gamma$  after the first time it hits  $B(z_0, e^{-I})$  is, absolutely continuous with respect to the unconditional law  $(\gamma_t, t \geq T_i)$  where  $T_i$  is the first time the path enters  $B(z_0, e^{-i})$ . Also the derivative of the conditional law with respect to the unconditional law is bounded by  $C > 0$ . Since the distance  $d$  is a continuous function of  $\gamma$  the estimate eq. (4.19) applies with  $\kappa = 2$  for all  $\delta$  sufficiently small and  $S - I$  large enough.  $\square$

Now we can finally prove our coupling estimate. Recall the full coupling  $(\gamma, \tilde{\gamma})$  of Lemma 4.18, where  $\gamma$  is a loop-erased random walk in  $D^{\#\delta}$  and  $\tilde{\gamma}$  is a loop-erased random walk in  $\tilde{D}^{\#\delta}$  starting from a vertex  $v$  where  $\tilde{D}$  is arbitrary. It is recommended to think of  $\tilde{D}$  as the full plane.

**Lemma 4.28.** *There exists a universal constant  $c$  such that the following holds. For any  $t > 0$  there exists  $\delta = \delta(t)$  such that for any  $\delta \in (0, \delta(t))$ , we can find a pair of random variables  $(X, \tilde{X})$  such that individually,  $X$  and  $\tilde{X}$  are each independent of  $(\gamma, \tilde{\gamma})$  and*

$$\mathbb{P}[\gamma(t + X) = \tilde{\gamma}(t + \tilde{X})] \geq 1 - Ce^{-ct}.$$

Furthermore, both  $X$  and  $\tilde{X}$  are random variables which are bounded (by  $1/20$ ).

In this proof, we will need to choose constants depending on each other so we number them.

*Proof.* Let  $\mu$  be the measure on  $\gamma$  defined as in Lemma 4.26 and let  $\tilde{\mu}$  be the equivalent measure for the curve  $\tilde{\gamma}$ . Note that Lemma 4.26 allows us to compare  $\mu$  to  $\bar{\mu}$  and also  $\tilde{\mu}$  to  $\bar{\mu}$  since  $\gamma$  and  $\tilde{\gamma}$  are assumed to coincide within  $B(z_0, e^{-I})$ . Consequently, we will have a way of comparing  $\mu$  and  $\tilde{\mu}$  and show that the Radon-Nikodym derivative of one with respect to the other is very close to one, from which the result will follow.

Here are the details. Let  $z = z(t)$  be the point of capacity  $t$  seen from  $v$  in  $D$  for  $\gamma$ . Let  $\tilde{T}(z)$  be the capacity of  $\tilde{\gamma}$  at  $z$  seen from  $v$  in  $\tilde{D}$ . Let  $\mathcal{A}$  be the event that  $I \leq t/2$ ,  $|\tilde{T}(z(t)) - t| \leq e^{-c_1 t}$  and

both  $\gamma, \tilde{\gamma}$  do not exit  $B(z_0, e^{-t/2})$  after capacity  $t$ . Recall that  $I$  has exponential tail (Theorem 4.19) and for small enough  $\delta = \delta(t)$ , we see from Lemma 4.24 and Schramm's estimate (3.3) that both  $\gamma, \tilde{\gamma}$  do not exit  $B(z_0, e^{-I})$  after capacity  $t$  with exponentially high probability. Thus overall, the probability of  $\mathcal{A}$  is at least  $1 - e^{-c_2 t}$  for some  $c_2 > 0$  for small enough  $\delta(t)$ .

Let  $d, a, A$  be as in Lemma 4.26 for  $\gamma, D$  and let  $\tilde{d}, \tilde{a}, \tilde{A}$  be the equivalent quantities for  $\tilde{\gamma}, \tilde{D}$ . By eq. (4.20), we observe that the event  $\mathcal{B} := \{d > Ae^{-a/4t}, \tilde{d} > \tilde{A}e^{-\tilde{a}/4t}\}$  has probability at least  $1 - e^{-c_3 t}$  for some  $c_3 > 0$ .

Applying Lemma 4.26, we find that for  $t$  sufficiently large on  $\mathcal{A} \cap \mathcal{B}$ ,

$$1 - e^{-c_4 t} \leq \frac{d\mu}{d\tilde{\mu}} \leq 1 + e^{-c_4 t} \quad (4.21)$$

on a subset of the path including  $\tilde{\gamma}(t + e^{-c_1 t}, t - e^{-c_1 t} + \frac{1}{20})$  for some small enough  $\delta(t)$ . Define  $\Gamma = \tilde{\gamma}(t + e^{-c_1 t}, t - e^{-c_1 t} + \frac{1}{20})$ . We let  $\nu = (1 - e^{-c_4 t})1_\Gamma \tilde{\mu}$ . Now we choose  $0 < c_5 < c_4$  and we define three independent Poisson processes on  $\gamma$ , say  $P_1, P_2, P_3$ , with densities respectively  $e^{c_5 t} \nu$ ,  $e^{c_5 t}(\tilde{\mu} - \nu)$  and  $e^{c_5 t}(\mu - \nu)$ . Note also that the processes  $P_1$  and  $P_2$  depend only on  $\tilde{\gamma}$ .

Let  $\tilde{z}$  be the first point in  $P_1 \cup P_2$  after capacity  $t$  in  $\tilde{D}$  and let  $z$  be the first point in  $P_1 \cup P_3$  after capacity  $t$  in  $D$ . Let  $X = T(z) - t$  and  $\tilde{X} = T(\tilde{z}) - t$ ; if there are no points in  $P_1 \cup P_3$  or  $P_1 \cup P_2$  we abort the coupling (this has probability smaller than  $e^{-\exp(c_5 t/30)}$ ). Note that  $X, \tilde{X} \leq 1/20$  by construction and they are each independent of  $(\gamma, \tilde{\gamma})$  (since the conditional marginal law of both  $X$  and  $\tilde{X}$  is exponential; however, note that the pair  $(X, \tilde{X})$  is *not* independent from  $(\gamma, \tilde{\gamma})$ ). Now observe that with very high probability both  $z, \tilde{z} \in P_1$  and therefore are equal. Indeed,  $\mathbb{P}(\tilde{z} \notin P_1) \leq e^{(c_5 - c_4)t}$  and similarly  $\mathbb{P}(z \notin P_1) \leq e^{(c_5 - c_4)t}$  using eq. (4.21) which concludes the proof.  $\square$

## 5 Convergence of the winding of uniform spanning tree to GFF

### 5.1 Discrete winding, definitions and notations

In this section we prove our main result on the winding of spanning tree.

We define the discrete winding fields in a finite domain properly here. This is completely analogous to the definition in the continuum from Section 3. Let us fix a bounded domain  $D \subset \mathbb{C}$  with a locally connected boundary and a marked point  $x \in \partial D$ . Using [37] Theorem 2.1,  $\partial D$  is a curve. Let  $\delta > 0$  and let  $\mathcal{T}^{\#\delta}$  be a wired spanning tree of  $D^{\#\delta}$ . Recall the definitions of  $\gamma_v^{\#\delta}$  and auxiliary vertices from Section 4.2. As in the continuum definition, we add to each  $\gamma_v^{\#\delta}$  a path following  $\partial D^{\#\delta}$  clockwise to  $x$ . More precisely, one endpoint of  $\gamma_v$  is some auxiliary vertex. We add to  $\gamma_v$  the *continuum* curve joining this vertex and  $x$  in the clockwise direction. For simplicity of notation we still call this path  $\gamma_v$  or  $\gamma_v^{\#\delta}$  to emphasise that this is a discrete path.

For our application to hexagonal lattice and  $T$ -graphs, we can also assume that the boundary of  $D^{\#\delta}$  consists of a simple oriented loop and there exists a marked vertex  $x^{\#\delta}$  in the boundary which converges to  $x$ . The fact that we can assume this is justified in [4]. In such a scenario, we can define  $\gamma_v$  to be the oriented path starting from  $v$  until it hits the boundary and then continuing along the boundary until we reach  $x^{\#\delta}$ .

We parametrise the part of the curve in  $\partial D$  by  $[-1, 0]$  and the rest by capacity in  $D$  minus  $\log R(z, D)$  as in Section 3), so that  $t = 0$  will always correspond to hitting the boundary. Motivated

by the formula connecting intrinsic and topological winding (cf. eq. (2.8)), we define

$$h^{\#\delta}(v) := W_{\text{int}}(\gamma_v(-1, \infty)) \quad (5.1)$$

$$h_t^{\#\delta}(v) := W(\gamma_v[-1, t], z) - \text{Arg}(-\gamma'_v(-1)) + \arg_{x-D}(x - v) \quad (5.2)$$

If  $D$  is not smooth near  $x$ , we define

$$h_t^{\#\delta}(v) := W(\gamma_v[-1, t], z) - \arg_{x-D}(x - v) \quad (5.3)$$

where  $\arg_{x-D}(x - v)$  is defined up to a global constant.

Consider a full plane discrete UST on  $G^{\#\delta}$ ,  $\tilde{\mathcal{T}} = (\tilde{\gamma}_v)_{v \in G^{\#\delta}}$ . We parametrise the paths  $\tilde{\gamma}_v$  by full plane capacity, going from  $-\infty$  far away to  $+\infty$  at  $v$ . Recall that we denote by  $\gamma(S, T)$  the path between capacities  $S$  and  $T$ . We extend the definition of the regularised winding to that setting by defining  $\tilde{h}_T(v) - \tilde{h}_S(v) := W(\tilde{\gamma}_v(S, T), v)$  and  $\tilde{h}(v) - \tilde{h}_T(v) := W(\tilde{\gamma}_v(T, \infty), v)$ . Note that the definition of the increments (in  $T$ ) of  $\tilde{h}$  make sense even though we cannot define  $\tilde{h}$  pointwise, so this definition is a slight abuse of notation. Finally we define

$$m^{\#\delta}(v) := \mathbb{E}[\tilde{h}^{\#\delta}(v) - \tilde{h}_0^{\#\delta}(v)]. \quad (5.4)$$

We point out that as  $\delta \rightarrow 0$ , the path  $\tilde{\gamma}_v$  converges (uniformly on compacts) towards a full plane SLE<sub>2</sub>; this follows by a combination of results from Lawler, Schramm and Werner; Yadin and Yehudayoff [31, 43] as well as Field and Lawler [11]. As a consequence, the arbitrary choice of truncating at capacity 0 is irrelevant: this is because the asymptotics of  $m^{\#\delta}(v)$  as  $\delta \rightarrow 0$  is independent of the choice of truncation (indeed, for a full plane SLE path, the expected winding between capacity 0 and 1 is zero by symmetry). Readers who are uncomfortable with the notion of full plane SLE can replace the full plane by a disc of some large radius in the definition of  $m^{\#\delta}$ ; in which case the above remark relies just on the convergence result of Lawler, Schramm and Werner [31] and Yadin and Yehudayoff [43].

To help with the intuition, recall from the introduction (see Figure 4) that we need to remove by hand microscopic contributions to the expected winding. This is the purpose of  $m^{\#\delta}$ ; note in particular as discussed in the introduction that this does not depend on the domain  $D$ . Note that we need to take into account possible contribution of intermediate scale between the microscopic and macroscopic one which is taken care of by subtracting  $m^{\#\delta}$ .

## 5.2 Statement of the main result

We now state the main theorem in this section, which is a stronger version of Theorem 1.2 in the introduction. Since we are going to integrate the discrete winding field against test functions, we need to make precise what we mean by this. There are two natural choices to do this integral: one which takes into account the geometry of the underlying graph and another one which accounts only for the ambient Euclidean space in which the graph is embedded. The latter turns out to be slightly more natural since the limit in that case is a (conformally invariant) Gaussian free field, i.e., does not depend on the limiting density of vertices.

We proceed as follows. Given  $h^{\#\delta}$  defined on the vertices of the graph, we can extend  $h^{\#\delta}$  to a function on the whole domain  $D$  using various forms of interpolation. One way to do this is to linearly extend the value of  $h^{\#\delta}$  to the edges and then take a harmonic extension on the faces (this includes the outer face, and then we restrict this extension to  $D$ ). However for concreteness we will

look at the following extension: consider the Voronoi tessellation of  $D$  defined by the vertices of the graph. We then define the extension  $h_{\text{ext}}^{\#\delta}$  to be constant on each Voronoi cell, equal to  $h^{\#\delta}(v)$  where  $v$  is the centre of the cell. This allows us to use the regular  $L^2$  product to integrate  $h^{\#\delta}$  against test functions.

$$(h^{\#\delta}, f) := \int_D h_{\text{ext}}^{\#\delta}(z) f(z) dz.$$

This extension procedure can also be applied to  $m^{\#\delta}$ , leading to a function defined on all of  $D$ . We then have the following theorem.

**Theorem 5.1.** *Let  $G$  be a graph satisfying the assumptions of Section 4.1, let  $D \subset \mathbb{C}$  be a simply connected domain with a locally connected boundary and a marked point  $x \in \partial D$ . Let  $\mathcal{T}$  be a uniform spanning tree of  $D^{\#\delta}$  and let  $h^{\#\delta}(v)$  denote the intrinsic winding as above, and let  $m^{\#\delta}$  be defined as above. Then*

$$h^{\#\delta} - m^{\#\delta} \xrightarrow[\delta \rightarrow 0]{} h_{\text{GFF}}.$$

*The convergence is in law in the Sobolev space  $H^{-1-\eta}$  for any  $\eta > 0$ . The limit  $h_{\text{GFF}}$  is a free field with intrinsic winding boundary conditions: that is,  $h_{\text{GFF}} = 1/\chi h + \pi/2$  where  $h = h_{\text{GFF}}^0 + \chi u_{(D,x)}$  and  $h_{\text{GFF}}^0$  is a standard GFF with Dirichlet boundary conditions and  $u_{(D,x)}$  is defined as in eq. (2.9) and Remark 2.5.*

*Moreover, for all  $n \geq 1$ , for all  $f_1, \dots, f_n \in H^{1+\eta}$ , and for all  $k_1, \dots, k_n$  we have*

$$\mathbb{E} \prod_i (h^{\#\delta} - m^{\#\delta}, f_i)^{k_i} \rightarrow \mathbb{E} \prod_i (h_{\text{GFF}}, f_i)^{k_i}$$

*and for all  $k \geq 1$*

$$\mathbb{E} \left( \|h^{\#\delta} - m^{\#\delta}\|_{H^{-1-\eta}}^k \right) \xrightarrow[\delta \rightarrow 0]{} \mathbb{E} \left( \|h_{\text{GFF}}\|_{H^{-1-\eta}}^k \right).$$

**Remark 5.2.** We emphasise that the function  $m^{\#\delta}$  is a deterministic function which depends only on the point in the graph, and in particular it does not on the domain  $D$ . Note also that it follows from this result that  $\mathbb{E}(h^{\#\delta} - m^{\#\delta}) \rightarrow \mathbb{E}(h_{\text{GFF}})$  and hence we deduce

$$h^{\#\delta} - \mathbb{E} h^{\#\delta} \rightarrow \frac{1}{\chi} h_{\text{GFF}}^0$$

in the same sense as above, where  $h_{\text{GFF}}^0$  is a Gaussian free field with Dirichlet boundary conditions.

### 5.3 Other notions of integration

We now comment briefly on other possible definitions of integration against test functions. Another definition which is a priori natural is to consider

$$(h^{\#\delta}, f)_{\#\delta} := \frac{1}{\mu^{\#\delta}(D)} \sum_v h^{\#\delta}(v) f(v).$$

In that case,  $h^{\#\delta}$  is viewed as a random measure which is a sum of Dirac masses and there is no hope to get convergence in the strong sense of Sobolev spaces as in the above theorem. We can nevertheless ask about convergence in the space of distributions (for finite dimensional marginals,



see e.g. [3, Definition 1.10]). It can be checked that if the uniform distribution on the vertices of the graph converges to a measure  $\mu$  in  $\mathbb{C}$ , we have that

$$(h^{\#\delta}, f)_{\#\delta} \xrightarrow[\delta \rightarrow 0]{} h_{\text{GFF}}^\mu$$

where now  $h_{\text{GFF}}^\mu$  is a Gaussian stochastic process indexed by test functions such that  $(h_{\text{GFF}}^\mu, \phi) = (h_{\text{GFF}}, \phi \frac{d\mu}{d\text{Leb}})$ .

Note that in most cases, for example in any periodic lattice or isoradial graphs or T-graphs where our results apply,  $\mu$  is just the Lebesgue measure. However there are also interesting examples of graphs where the convergence to a (time-changed) Brownian motion holds but  $\mu$  is different from Lebesgue measure. For instance, for a conformally embedded random planar map, where such a convergence is expected to hold, the measure  $\mu$  is a Gaussian multiplicative chaos (see [12] and [13, 2] for an introduction to this topic).

## 5.4 Proof of the main result

Now we collect the results of the two previous sections to prove Theorem 5.1.

**Lemma 5.3.** *Fix a domain  $D \subset \mathbb{C}$  with locally connected boundary and let  $x_1, \dots, x_k \in D$ . For all  $v_1^\delta, \dots, v_k^\delta \in D^{\#\delta}$  converging to  $x_1, \dots, x_k$ , for all  $T_1, \dots, T_k$ ,*

$$\mathbb{E}[\prod_i h_{T_i}^{\#\delta}(v_i)] \xrightarrow[\delta \rightarrow 0]{} \mathbb{E} \prod_i h_{T_i}(x_i).$$

where  $h_T$  is the regularised winding field of a continuum UST in  $D$  as defined in eq. (2.9) and Remark 2.5.

*Proof.* By our assumptions (see Remark 4.1) and by Wilson's algorithm, the paths  $(\gamma_{v_i}^{\#\delta})_{i=1}^k$  converge to  $(\gamma_{x_i})_{i=1}^k$  where the  $\gamma_{x_i}$  are the paths connecting  $x_i$  to  $\partial D$  in a continuous UST. Furthermore, observe that the function  $h_T(v)$  is a continuous function of  $\gamma_v^{\#\delta}$  (this is because the topological winding up to capacity  $t$  is continuous in the curve). Hence  $\prod h_T^{\#\delta}(v_i)$  converges in distribution to  $\prod h_T(x_i)$ . Using Proposition 4.10, we see that it is also uniformly integrable (in fact, we have stretched exponential tails) and hence the expectation converges.  $\square$

Combining the above lemma with Theorem 3.2 and Lemma 3.15, we can find a sequence  $T(\delta)$  (depending on the  $v_i$ 's) going to infinity slowly enough such that whenever  $T(\delta) \leq T_i(\delta) \leq T(\delta) + 1/20$ ,

$$\mathbb{E}[\prod h_{T_i(\delta)}^{\#\delta}(v_i)] \xrightarrow[\delta \rightarrow 0]{} \mathbb{E} \prod h_{\text{GFF}}(x_i).$$

Using the previous lemma with the full coupling of Theorem 4.19, we can control the  $k$ -point function for the winding up to the endpoint.

**Proposition 5.4.** *For all  $k$ , for all domains  $D$  with locally connected boundary, for all  $v_1^\delta, \dots, v_k^\delta \in D$  converging to  $x_1, \dots, x_k$ ,*

$$\mathbb{E}[\prod_i (h^{\#\delta}(v_i^\delta) - m^{\#\delta}(v_i^\delta))] \xrightarrow[\delta \rightarrow 0]{} \mathbb{E} \prod_i h_{\text{GFF}}(x_i).$$

Recall that the right hand side is a notation for the  $k$ -point function of a GFF with winding boundary condition, as in Theorem 5.1.

*Proof.* By definition of our extension of  $h^{\#\delta}$  to  $D$ , we may assume without loss of generality that  $v_i^\delta \in D^{\#\delta}$  (this is one of the advantages of working with the Voronoi extension of  $h^{\#\delta}$  to  $D$ ). We write  $\gamma_i$  for  $\gamma_{v_i}^{\#\delta}$ . If  $\delta$  is small enough, one can apply the coupling of Theorem 4.19. Focusing only on the paths from the  $v_i$ , we obtain random variables  $I_1, \dots, I_k$ , all with exponential tails, and independent full plane loop-erased paths  $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$  such that

$$\forall j, \quad \gamma_j \cap B(v_j, 6^{-I_j}r) = \tilde{\gamma}_j \cap B(v_j, 6^{-I_j}r)$$

on the event that the coupling succeeds, where  $r = (1/10)(\min_{i \neq j} |v_i - v_j| \wedge \min_i d(v_i, \partial D) \wedge 1)$ . Here  $r$  is a constant and  $\delta \rightarrow 0$  so we will not worry about  $r$ . Let  $\tilde{h}^{\#\delta}$  be the associated field defined as in eq. (5.4) (recall that only its increments in  $T$  are defined). We also write  $h_I^{\#\delta} = h_I^{\#\delta}(v_j)$  (resp.  $\tilde{h}_I^{\#\delta}$ ) for the winding up to the last time  $\gamma_j$  (resp.  $\tilde{\gamma}_j$ ) enters  $B(v_j, e^{-I_j})$ . Let  $T(\delta)$  be some sequence such that for any  $T_j$  such that  $T(\delta) \leq T_j \leq T(\delta) + 1/20$  we have

$$\mathbb{E}[\prod h_{T_j}^{\#\delta}(v_j)] \rightarrow \mathbb{E}[\prod h_{\text{GFF}}(x_j)], \quad (5.5)$$

$$\mathbb{E}[\tilde{h}_{T(\delta)}^{\#\delta}(v_j) - \tilde{h}_0^{\#\delta}(v_j)] \rightarrow 0 \quad (5.6)$$

and Lemma 4.28 holds for  $t = T(\delta)$ . This is possible because  $T(\delta)$  can be chosen to grow to infinity arbitrarily slowly and the fact that the branch from  $v_j$  to infinity converges to a whole plane SLE<sub>2</sub> which is symmetric with respect to conjugation (this is the same reason why  $m^{\#\delta}(v)$  does not depend on the capacity cutoff, see the discussion after eq. (5.4)).

For each  $1 \leq j \leq k$  we use Lemma 4.28 and we write  $T_j = T(\delta) + X_j$  and  $\tilde{T}_j = T(\delta) + \tilde{X}_j$  the resulting times. Then we have, dropping the superscripts  $^{\#\delta}$  from this calculation for clarity,

$$h(v_j) - m(v_j) = (h - h_I)(v_j) - (\tilde{h} - \tilde{h}_I)(v_j) \quad (A_j)$$

$$+ (h_I - h_{T_j})(v_j) - (\tilde{h}_I - \tilde{h}_{\tilde{T}_j})(v_j) \quad (B_j)$$

$$- (\tilde{h}_{\tilde{T}_j}(v_j) - \tilde{h}_0(v_j)) \quad (C_j)$$

$$+ \tilde{h}(v_j) - \tilde{h}_0(v_j) - m(v_j) \quad (D_j)$$

$$+ h_{T_j}(v_j). \quad (\xi_j)$$

The main term here is the last one,  $h_{T_j}(v_j)$ , which we will call  $\xi_j$ : from eq. (5.5), it follows that  $\mathbb{E} \prod_j \xi_j$  converges to  $\mathbb{E} \prod_i h_{\text{GFF}}(x_i)$ . All the other terms will be small in the sense of moments. Let  $A_j, B_j, C_j, D_j$  be the random variables in  $(A_j), (B_j), (C_j), (D_j)$  respectively and let  $\xi_j = h_{T_j}(v_j)$ .

Note that  $A_j$  is identically zero if the coupling succeeds, by definition of the coupling and eq. (2.8). Note also that  $B_j = 0$  if the coupling succeeds and  $T_j \geq I_j$ . We expand the product

$$\prod_j (A_j + B_j + C_j + D_j + \xi_j) = \sum \prod_j \mathcal{L}_j$$

where for each  $j$ , the product above is over  $\mathcal{L}_j \in \{A_j, B_j, C_j, D_j, \xi_j\}$ . We now summarise the key properties which allow us to deal with this product.

- (i)  $\mathbb{E}(|A_i|^k) \leq O(\log(1/\delta)^k \delta^\alpha)$ . (Corollary 4.20).
- (ii)  $\mathbb{E}(|B_j|^k) \leq O(T_j^k e^{-cT_j^\alpha})$ . (Proposition 4.10 and Theorem 4.19).

- (iii)  $\mathbb{E}(C_j) = o(1)$  and  $C_j$  is independent from  $(\mathcal{L}_i, i \neq j)$ . (eq. (C<sub>j</sub>)).
- (iv)  $\mathbb{E}(D_j) = 0$  and  $D_j$  is independent from  $(\mathcal{L}_i, i \neq j)$ . (Definition of  $m$ .)
- (v)  $\mathbb{E}(|\xi_j|^k) \leq O(T_j^k)$  and  $\mathbb{E}(\prod_j \xi_j) = O(1)$  where the product is over any subset of indices. (Proposition 4.10 and eq. (5.5)).

Hence any product involving  $D_j$  for some  $j$  will contribute zero to the expectation. If the product involves only of the terms of the form  $A_j, B_j, \xi_j$  and there is at least one  $A_j$  or  $B_j$ , then the contribution to the expectation is  $o(1)$  by Hölder. For terms involving only  $C_j$  and  $\xi_j$ , the  $C_j$  factors out and we conclude using item (v) above. For the remaining terms, the  $C_j$  also factors out and we conclude as above by Hölder.

We now explain the claims above. For item (i), note that  $h(v_j) - h_I(v_j)$  only depends on  $\gamma_j \cap B(v_j, e^{-I_j})$ . So the first term is exactly 0 by construction except in the low probability event of aborting the full coupling. This happens with a probability  $O(\delta^\alpha)$  for some power  $\alpha > 0$  by Corollary 4.20. On the other hand, note that for any  $C$ ,  $h_{-\log(\delta)-C}(v_j) - h(v_j)$  is bounded by the bounded density and bounded winding assumptions on the graph (i and ii). For example, the total winding in a ball can be bounded by the number of vertices in it. Furthermore if  $C$  is chosen large enough, Proposition 4.10 applies for  $h_t$  up to to capacity  $t = -\log(\delta) - C$  and hence we conclude that the  $p$ th moment of  $h_{\log \delta - C}(v_j) - h_I$  is  $O(\log^p \delta)$ . The same bounds also holds for  $\tilde{h}(v_j) - \tilde{h}_I(v_j)$  so we conclude using Cauchy–Schwarz on the low probability event.

Item (ii) is very similar.  $B_j$  is exactly 0 when the coupling succeeds and  $T(\delta) \geq I_j$  which happens with probability  $1 - O(e^{-cT(\delta)})$ . On the other hand, observe that  $h_I - h_{T_j}$  is stochastically dominated by a sum of  $|I - T_j|$  terms with stretched exponential tail again by Proposition 4.10, and  $|I - T_j|$  has exponential tail. We conclude as above by Cauchy–Schwarz and Lemma 4.9.

We now turn to item (iii) which deals with  $C_j$ . Note that  $C_j$  involves only  $\tilde{\gamma}_j$  and  $\tilde{X}_j$  and so by construction of the full coupling this will be independent of  $(\mathcal{L}_i, i \neq j)$ . Furthermore,  $\mathbb{E}(C_j) \rightarrow 0$  by eq. (5.6) and since  $\tilde{X}_j$  is independent of  $\tilde{\gamma}_j$  by construction.

For item (iv), the independence holds exactly for the same reason as argued above for  $C_j$ . The fact that  $\mathbb{E}(D_j) = 0$  is by definition of  $m(v_j)$ .

Finally, for item (v), the moment bound on  $\xi_j$  is a consequence of Proposition 4.10, while the fact that  $\mathbb{E}(\prod_j \xi_j) = O(1)$  comes directly from eq. (5.5). This concludes the proof of the lemma.  $\square$

The above proposition gives a pointwise convergence of the  $k$ -point function. We now need some a priori bounds to allow us to integrate this convergence of moments against test functions and use the dominated convergence theorem.

**Lemma 5.5.** *For all  $k \geq 2$ , for all domains  $D$  with locally connected boundary, there exist constants  $C = C_k, c > 0$  such that for all  $\delta < c\delta_0$ , for all  $v_1^{\#\delta}, \dots, v_k^{\#\delta} \in D^{\#\delta}$ ,*

$$\left| \mathbb{E} \left[ \prod (h^{\#\delta}(v_i^{\#\delta}) - m^{\#\delta}(v_i^{\#\delta})) \right] \right| \leq C(1 + \log^{4k} r)$$

where  $r = (1/10)(\min_{i \neq j} |v_i^{\#\delta} - v_j^{\#\delta}| \wedge \min_j d(v_j^{\#\delta}, \partial D) \wedge 1)$ . The same holds even if we replace  $v_i^{\#\delta}$  by any point in its Voronoi cell as in our extended definition of  $h^{\#\delta}(z)$ .

*Proof.* Let us assume  $r \geq \delta$  for now. Consider the full coupling of Theorem 4.19. Then we have variables  $I_j$  with exponential tails and independent full plane loop-erased path  $\tilde{\gamma}_j$  such that

$$\gamma_j \cap B(v_j, 6^{-I_j} r) = \tilde{\gamma}_j \cap B(v_j, 6^{-I_j} r)$$

on the event that the coupling succeeds.

We use a decomposition similar as in the previous proof (except that we do not need  $h_{T_j}$  or  $\tilde{h}_{\tilde{T}_j}$ ). For clarity let us drop the superscript  $\# \delta$  in the following expression:

$$\begin{aligned}
h - m &= (h - h_I) - (\tilde{h} - \tilde{h}_I) & (A'_j) \\
&+ h_I & (B'_j) \\
&- (\tilde{h}_I - \tilde{h}_0) & (C'_j) \\
&+ \tilde{h} - \tilde{h}_0 - m. & (D'_j)
\end{aligned}$$

It suffices to check that the moments of order  $k$  are bounded by  $C(1 + (\log r)^{4k})$ . Let  $A'_j, \dots, D'_j$  respectively be the four terms in the right hand side above.

Let  $F$  be the event that the coupling fails. The first term is exactly 0 on the event that the coupling succeeds. As above on the event that the coupling fails, we can use Cauchy-Schwarz, Proposition 4.10 and Corollary 4.20 to get

$$\left| \mathbb{E}[(A'_j)^k 1_F] \right| \leq (\log r / \delta) (\delta / r)^\alpha \leq C$$

The second term  $B'_j$ , which is the main term, is controlled as follows. Note that

$$|h_I| \leq \sum_{i=-\log(R(v_j, D)/r)}^I w_i$$

where the  $i$ th term accounts for the winding in the annulus  $A(v_j, e^{-i}r, e^{(-i+1)r})$ . Now each of these terms have stretched exponential tail by Proposition 4.10 and hence the above sum from  $-\log(R(v_j, D)/r)$  to 0 is bounded by  $C(1 + (\log r)^k)$ . Further,  $I$  itself has exponential tail (independent of  $r$  and  $\delta$ ) by eq. (4.8) in Theorem 4.19. Using Lemma 4.9 to control the part of the sum corresponding to  $i = 1$  to  $I$ , we deduce that  $\mathbb{E}(|h_I|^k) \leq C(1 + (\log r)^k)$ . Note also that this argument works exactly in the same way for  $C'_j$ , replacing  $h_I$  by  $\tilde{h}_I - \tilde{h}_0$ .

As for the last term  $D'_j$ , it is independent from  $(\mathcal{L}'_i, i \neq j)$  where  $\mathcal{L}'_j \in \{A'_j, \dots, D'_j\}$  and  $\mathbb{E}(D'_j) = 0$ . Therefore when we expand the product  $\mathbb{E}(\prod_j (A'_j + \dots D'_j))$  every term which includes at least one  $D'_j$  will contribute zero to the expectation. The other terms will contribute at most  $C(1 + (\log r)^{4k})$  by Hölder.

Finally if  $r < \delta$ , we can use Hölder to bound the moment of  $h^{\# \delta} - m^{\# \delta}$  by  $C(1 + \log^{4k} \delta)$  as above which is at most the required bound.  $\square$

We can now prove our main theorem.

*Proof of Theorem 5.1.* Fix  $f_1, \dots, f_n$  to be smooth functions in  $\bar{D}$  and  $k_1, \dots, k_n \geq 1$ . Combining Proposition 5.4 and Lemma 5.5, we see that we can apply the dominated convergence to  $\mathbb{E} \left[ \prod_{i=1}^n (h^{\# \delta} - m^{\# \delta}, f_i)^{k_i} \right]$  and therefore

$$\mathbb{E} \left[ \prod_{i=1}^n (h^{\# \delta} - m^{\# \delta}, f_i)^{k_i} \right] \rightarrow \mathbb{E} \left[ \prod_{i=1}^n (h_{\text{GFF}}, f_i)^{k_i} \right].$$

Hence, since the right hand side is Gaussian (and therefore moments characterise the distribution),  $(h^{\# \delta} - m^{\# \delta}, f_i)_{i=1}^n$  converges in distribution to  $(h_{\text{GFF}}, f_i)_{i=1}^n$ . In other words, at this point we

already know  $h^{\# \delta} - m^{\# \delta}$  converges to  $h_{\text{GFF}}$  in the sense of finite dimensional marginals (when viewed as a stochastic process indexed by smooth functions with compact support, say). We now check tightness in the Sobolev space  $H^{-1-\eta}$ , from which convergence in  $H^{-1-\eta}$  will follow.

Note that by the Rellich–Kondrachov embedding theorem, to get tightness in  $H^{-1-\eta}$  it suffices to prove that  $\mathbb{E}(\|h^{\# \delta} - m^{\# \delta}\|_{H^{-1-\eta'}}^2) < C$  for some constant  $C$ , for any  $\eta' < \eta$ . More generally we will check that  $\mathbb{E}(\|h^{\# \delta} - m^{\# \delta}\|_{H^{-1-\eta'}}^{2k}) < C_k$  for any  $k \geq 1$ . Without loss of generality we write  $\eta$  for  $\eta'$ .

Let  $(e_i)$  be an orthonormal basis of eigenfunctions of  $-\Delta$  in  $L^2(D)$ , corresponding to eigenvalues  $\lambda_j > 0$ . Then writing  $h = h^{\# \delta} - m^{\# \delta}$  for convenience,

$$\mathbb{E}(\|h\|_{H^{-1-\eta}}^{2k}) = \mathbb{E} \left( \sum_{j=1}^{\infty} (h, e_j)_{L^2}^2 \lambda_j^{-1-\eta} \right)^k \leq C \sum_{j=1}^{\infty} \mathbb{E}((h, e_j)_{L^2}^{2k}) \lambda_j^{-1-\eta}$$

by Fubini's theorem and Jensen's inequality (since by Weyl's law,  $\sum_j \lambda_j^{-1-\eta} < \infty$  is summable).

Furthermore,

$$\mathbb{E}((h, e_j)_{L^2}^{2k}) = \int_{D^{2k}} \mathbb{E}(h(z_1) \dots h(z_{2k})) e_j(z_1) \dots e_j(z_{2k}) dz_1 \dots dz_{2k}$$

and note that by Lemma 5.5,  $\mathbb{E}(h(z_1) \dots h(z_{2k})) \leq C(1 + \log^{8k} r)$  where  $r = r(z_1, \dots, z_{2k})$  is as in that lemma. Note also that  $(\log r)^a$  is integrable for any  $a > 0$  and  $D$  is bounded hence using Cauchy–Schwarz, we deduce that  $\mathbb{E}((h, e_j)_{L^2}^{2k}) \leq C \int_{D^{2k}} e_j(z_1)^2 \dots e_j(z_{2k})^2 dz_1 \dots dz_{2k} = C$  by Fubini and since  $e_j$  is orthonormal in  $L^2$ . Consequently,

$$\mathbb{E}(\|h\|_{H^{-1-\eta}}^{2k}) \leq \sum_{j=1}^{\infty} C \lambda_j^{-1-\eta} < \infty$$

by Weyl's law. This finishes the proof of Theorem 5.1 and hence Theorem 1.2. Let us remind the reader here that these proofs of moment bounds in  $H^{-1-\eta}$  follows through in exactly the same way in the continuum proof in Proposition 3.25.  $\square$

## 6 Joint Convergence

In this section we prove the joint convergence of the height function and the wired UST pair to the GFF and continuum wired UST pair where the latter is coupled together through the imaginary geometry coupling in Theorem 2.6.

Let us first introduce the setup. The topology on the height function coordinate is the Sobolev space  $H^{-1-\eta}$  as described in Proposition 2.4 (recall this is a complete, separable Hilbert space). The topology on the tree coordinate will be the Schramm topology  $\Omega_1$  introduced in [39] and defined in Section 2.5. As usual we have a bounded domain  $(D, x)$  with a marked point  $x \in \partial D$  and locally connected boundary. We work with the space  $\Omega := \Omega_1 \times H_{-1-\eta}$  equipped with the product topology. We also view  $\Omega$  also as a metric space with metric defined by  $d_1 + d_2$  where  $d_1$  and  $d_2$  are the metrics in each coordinate. Let  $\mathcal{T}^d, h^{\# \delta}$  be as in Section 5.2. Let  $(\mathcal{T}, h_{\text{GFF}}(\mathcal{T}))$  denote the continuum wired UST in  $D$  with  $h_{\text{GFF}}(\mathcal{T})$  denoting the GFF which is coupled with  $\mathcal{T}$  as described in Theorem 3.21. The point here is again that the height function is not continuous as a function of the discrete tree, hence we have to use the results about the truncated winding we proved in this paper.

**Theorem 6.1.** *In the above setup, we have the following joint convergence in law in the product topology described above:*

$$(\mathcal{T}^{\#\delta}, h^{\#\delta} - m^{\#\delta}) \xrightarrow[\delta \rightarrow 0]{(d)} (\mathcal{T}, h_{\text{GFF}}(\mathcal{T})).$$

*Proof.* To simplify notations, we write  $h^{\#\delta}$  for  $h^{\#\delta} - m^{\#\delta}$  admitting a slight abuse of notation. Let  $h_t$  denote the continuum winding truncated at capacity  $t$  as before. Notice that from Theorem 3.21,

$$(\mathcal{T}, h_t) \xrightarrow[t \rightarrow \infty]{P} (\mathcal{T}, h_{\text{GFF}}(\mathcal{T})).$$

where the above convergence is in probability in the metric space defined above. Since for any fixed  $t$ ,  $h_t^{\#\delta}$  is a continuous function of  $\mathcal{T}^{\#\delta}$  in  $\delta$  (with respect to the Schramm topology). Note that for any fixed  $t$ ,  $h_t^{\#\delta}$  and  $h_t$  are obtained by applying the same function to respectively  $\mathcal{T}^{\#\delta}$  and  $\mathcal{T}$ . Since this function is continuous when  $\mathcal{T}^{\#\delta}$  and  $\mathcal{T}$  are seen as variables in the Schramm space, we have, we have

$$(\mathcal{T}^{\#\delta}, h_t^{\#\delta}) \xrightarrow[\delta \rightarrow 0]{(d)} (\mathcal{T}, h_t)$$

Thus there exists a sequence  $t(\delta)$  growing slow enough such that

$$(\mathcal{T}^{\#\delta}, h_{t(\delta)}^{\#\delta}) \xrightarrow[\delta \rightarrow 0]{(d)} (\mathcal{T}, h_{\text{GFF}}(\mathcal{T}))$$

Using the equivalence of weak convergence and the convergence in Lévy–Prokhorov metric in a separable space, we obtain that for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all Borel set  $A$ ,

$$\mathbb{P}((\mathcal{T}^{\#\delta}, h_{t(\delta)}^{\#\delta}) \in A) < \mathbb{P}((\mathcal{T}, h_{\text{GFF}}(\mathcal{T})) \in A^\varepsilon) + \varepsilon \text{ and } \mathbb{P}((\mathcal{T}, h_{\text{GFF}}(\mathcal{T})) \in A) < \mathbb{P}((\mathcal{T}^{\#\delta}, h_{t(\delta)}^{\#\delta}) \in A^\varepsilon) + \varepsilon.$$

Now recall that we proved in Proposition 5.4 and Lemma 5.5 that  $\mathbb{E}(\|h_{t(\delta)}^{\#\delta} - h^{\#\delta}\|_{-1-\eta}) \rightarrow 0$  which implies that  $h_{t(\delta)}^{\#\delta} - h^{\#\delta}$  converges to 0 in probability. Thus we have for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\mathbb{P}((\mathcal{T}^{\#\delta}, h^{\#\delta}) \in A) < \mathbb{P}((\mathcal{T}, h_{\text{GFF}}(\mathcal{T})) \in A^{2\varepsilon}) + 2\varepsilon \text{ and } \mathbb{P}((\mathcal{T}, h_{\text{GFF}}(\mathcal{T})) \in A) < \mathbb{P}((\mathcal{T}^{\#\delta}, h^{\#\delta}) \in A^{2\varepsilon}) + 2\varepsilon$$

which completes the proof.  $\square$

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